

Gravitational Lensing by Rotating Braneworld Wormholes: Probing Extra-Dimensional Effects

(Pelensaan Gravitasi oleh Lohong Cecacing Berputar dalam Dunia Bran: Analisis Kesan Dimensi Tambahan)

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ABSTRACT

In this study, we investigate gravitational lensing by rotating wormholes in the braneworld framework. By deriving the deflection angle for a rotating wormhole spacetime with extra-dimensional contributions, we analyze how rotation and braneworld effects modify lensing signatures. The metric incorporates a rotational parameter ω , and the resulting deflection angle includes additional contributions arising from extra-dimensional braneworld corrections. Our results show, for the first time, the coupled contribution of rotation and braneworld corrections to the deflection angle, including an explicit dependence on the expansion factor, which is absent in previous rotating. These corrections distinguish braneworld wormholes from general relativistic analogs, offering new insights into distinguishing rotating wormholes from black holes and testing braneworld cosmology through gravitational lensing.

Keywords: Braneworld gravity; deflection angle; extra dimensions; gravitational lensing; rotating wormholes

ABSTRAK

Dalam penyelidikan ini, kami mengkaji kanta graviti oleh lohong cecacing berputar dalam kerangka dunia bran. Dengan menerbitkan sudut pesongan bagi ruang masa lohong cecacing berputar yang mempunyai sumbangan daripada dimensi tambahan, kami menganalisis bagaimana kesan putaran dan dunia bran mengubah ciri kanta graviti. Metrik yang digunakan menggabungkan suatu parameter putaran, manakala sudut pesongan yang terhasil merangkumi sumbangan tambahan yang timbul daripada pembetulan dunia bran berdimensi tambahan. Hasil yang diperolehi buat pertama kalinya, menunjukkan sumbangan gandingan antara putaran dan pembetulan dunia bran terhadap sudut pesongan, termasuk kebergantungan eksplisit kepada faktor pengembangan, yang tidak terdapat dalam kajian putaran terdahulu. Pembetulan ini membezakan lohong cecacing dunia bran daripada analognya dalam kerelatifan am, sekali gus menawarkan pandangan baharu dalam membezakan lohong cecacing berputar daripada lohong hitam serta menguji kosmologi dunia bran melalui kanta graviti.

Kata kunci: Dimensi tambahan; graviti dunia bran; kanta graviti; lohong cecacing berputar; sudut pesongan

INTRODUCTION

Gravitational lensing has emerged as an indispensable astrophysical tool for studying massive celestial structures, such as neutron stars, galaxies, supermassive black holes, and galaxy clusters (Koopmans et al. 2009; Saha et al. 2024; Smith et al. 2025). It enables high-precision mapping of matter distribution including elusive dark matter and grants deep insight into the geometry and expansion of the universe (Singh 2024). Moreover, it offers a unique means of testing the validity of general relativity (GR) on cosmological scales and exploring alternatives like modified gravity or extra-dimensional theories (Tizfahm et al. 2025). By generating multiple images of a single background

source (Schechter & Schnittman 2025), magnification (Palencia et al. 2024), and time delays (Birrer et al. 2024), gravitational lensing allows us to observe and analyze distant, otherwise inaccessible regions of the cosmos with remarkable accuracy.

The Randall-Sundrum (RS) type II braneworld cosmology framework (Sengupta 2024), which underpins our investigation, offers a robust and theoretically consistent model in which a four-dimensional flat Minkowski spacetime is dynamically embedded within a higher-dimensional bulk, enabling the exploration of extra-dimensional gravitational effects. This configuration facilitates dynamic interactions between the brane and the bulk, potentially showing imprints of

extra-dimensional physics through observable astrophysical phenomena. Foundational investigations of wormholes within this framework have laid the groundwork for more generalized models that probe how bulk-brane dynamics influence gravitational behavior (Wang, Wang & Wei 2025).

While static wormholes have been extensively studied as gravitational lenses (Teo 1998), achieving astrophysical realism requires the inclusion of rotation. Rotating wormholes not only generalize static solutions but may also stabilize geometries that would otherwise be unstable (Dzhunushaliev et al. 2013; Kuhfittig 2003), thereby potentially producing distinctive observational signatures. Within GR, rotation introduces frame-dragging effects, which change spacetime curvature and influence the trajectories of light rays (Bugaev et al. 2021; Jusufi & Övgün 2018). These effects are crucial for lensing phenomena, as they modify the deflection angle α and lead to asymmetric image distortions features absent in static spacetimes (Gyulchev et al. 2018).

In the braneworld, rotation further entangles with extra-dimensional physics. Bulk-brane interactions can amplify ω -dependent terms, introducing corrections to α that distinguish braneworld wormholes from their GR analogs (Pal & Kar 2008). Recent studies show that rotating wormholes supported by exotic fields like three-forms (Koivisto & Nunes 2010; Wongjun 2017), may lead to distinct geometric and observational features compared to their static counterparts, or, within modified gravity frameworks, can circumvent energy condition violations while producing detectable lensing anomalies (Nedkova, Tinchev & Yazadjiev 2013; Rahaman et al. 2021).

In this work, we construct a rotating wormhole metric within the braneworld framework and analyze its gravitational lensing characteristics. The novelty lies in incorporating both rotation and extra-dimensional corrections into the deflection angle. The paper is structured as follows: Next section presents the rotating wormhole metric, establishes its connection to gravitational lensing, and presents the full derivation of the deflection angle, demonstrating how braneworld dynamics influence the light-bending behavior around such exotic geometries.

WORMHOLES WITH ROTATING METRIC TO LENSING

From GR it can be showed that a light ray passing by a point mass M (ξ) will bend, or generally, a spherical mass distribution is deflected from a rectilinear motion by an angle $\alpha = 4GM(< \xi)/c^2\xi$, where M (ξ) is the mass enclosed within the cylinder parallel to the light ray, with the influence parameter (radius ξ). Brane cosmology, exemplified by the RS2 model, yields a flat Minkowski spacetime with brane-bulk interactions. Bronnikov and Kim (2003) pioneered the study of wormholes within this framework, while Lobo later introduced a more generalized class (Lobo 2007). To

derive the deflection angle near the lens, we begin by examining the null geodesic of light emitted from the source as it propagates under the influence of the curved spacetime characterized by the lens's geometry $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$ and using

$$ds^2 = A(x)dt^2 - B(x)dx^2 - C(x)d\theta^2 + \sin^2 \theta d\varphi^2, \quad (1)$$

where A , B and C are the metric components.

The closest approach of a light ray travelling toward the object responsible for its deflection is defined as $x_0 = \frac{r_0}{2M}$, where r_0 denotes the event-horizon radius in the black-hole case (Virbhadra & Ellis 2000) and M represents the mass of the object. The deflection angle at the lens can then be expressed as a function of x_0

$$\alpha(x_0) = I(x_0) - \pi, \quad (2)$$

where the integral term $I(x_0)$ represents the angular change experienced by the light ray near the deflecting object. To obtain the integral expression for the deflection angle in the vicinity of the lens, we begin by considering the null geodesic followed by light emitted from the source as it propagates through spacetime.

By applying the null geodesic condition $ds^2 = 0$, thus from Equation (1) we may write

$$A(x)dt^2 - C(x)d\theta^2 = 0, \quad (3)$$

$$-B(x)dx^2 - C(x)\sin^2 \theta d\varphi^2 = 0. \quad (4)$$

From Equations (3) and (4), we have

$$\left(\frac{d\theta}{dt}\right)^2 = \frac{A(x)}{C(x)}, \quad (5)$$

$$\left(\frac{d\varphi}{dx}\right)^2 = -\frac{B(x)}{C(x)\sin^2 \theta}. \quad (6)$$

From Equation (5), the lens impact parameter is defined as $u = 1/(d\theta/dt)$, which is the ratio of metric tensors $u = \frac{g_{22}/g_{00}}{\sqrt{C(x_0)/A(x_0)}}$. Thus, from Equation (5), we obtain $u = \sqrt{C(x)/A(x)}$. In the area nearer to the lens source, the influence parameter is stronger in the direction of the lens when $x \rightarrow x_0$ compared to that in a region just beyond the lens, that is, $u_0 > u_{\text{region} - \text{away}}$, and therefore $\sqrt{C(x_0)/A(x_0)} > \sqrt{C(x)/A(x)}$. Hence, Figure 1(a) shows how the radius near the lens compares with the radius slightly farther away with respect to the curvature angle θ , as light emitted by the source S approaches the lens L .

To simplify the geometric relation shown in Figure 1(b), we introduce l as the auxiliary geometric length corresponding to the segment determined by the difference between the metric-dependent path lengths near and away from the lens. Thus, we may write

$$l = \sqrt{C(x_0)A(x) - C(x)A(x_0)}. \quad (7)$$

From Figure 1(b) for $\sin \theta = l / \sqrt{C(x_0)A(x)}$. Squaring both sides gives $\sin^2 \theta = l^2 / C(x_0) A(x)$, substituting l^2 from Equation (7) and simplifying yields

$$\sin \theta = \frac{\sqrt{C(x_0)A(x) - C(x)A(x_0)}}{\sqrt{C(x_0)A(x)}} = \sqrt{1 - \frac{C(x)A(x_0)}{C(x_0)A(x)}}. \quad (8)$$

From Equations (6) and (8), considering $I(x_0) = 2\varphi$ where φ is the azimuthal angle parameterizing the photon trajectory in the equatorial plane. The integration can be given regarding the spacetime line element as (Tejeiro & Larrañaga 2005)

$$I(x_0) = \int_{x_0}^{\infty} \frac{2\sqrt{B(x)}dx}{\sqrt{C(x)} \sqrt{\left(\frac{C(x)}{C(x_0)}\right)\left(\frac{A(x_0)}{A(x)}\right) - 1}}. \quad (9)$$

For a Schwarzschild-type object, like a star, black hole, or massive planet, the line element reads

$$ds^2 = -(1 - 2Gm/r) dt^2 + (1 - 2Gm/r)^{-1} dr^2 + r^2 dr^2 + r^2 \sin^2 \theta d\phi^2, \quad (10)$$

where the redshift function is defined as $e^{2\Phi(r)} = 1 - 2Gm/r$. This relation is obtained directly from the temporal component of the Schwarzschild metric in Equation (10). Then, by using Equations (2), (9), and (10), the deflection angle can be expressed as

$$\alpha = 2 \int_{r_0}^{\infty} \frac{(1 - 2Gm/r)^{-1/2}}{r \sqrt{\frac{r^2}{u^2(1 - 2Gm/r)} - 1}} dr - \pi, \quad (11)$$

where u represents the gravitational lens impact parameter, which is defined as the ratio between the object's characteristic curvature scale r_0 (the radius of the black hole's event horizon) and its associated redshift function $\sqrt{(1 - 2Gm/r_0)}$, i.e., $u = r_0 / \sqrt{(1 - 2GM/r_0)}$. The analysis is broadened to include a wormhole spacetime described by Lemos, Lobo and de Oliveira (2003), as follows

$$ds^2 = -e^{2\Phi(r)} c^2 dt^2 + \frac{dr^2}{1 - b(r)/r} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \quad (12)$$

Comparing Equation (12) with the general spherically symmetric metric gives $A(r) = e^{2\Phi(r)}$, $B(r) = \left[1 - \frac{b(r)}{r}\right]^{-1}$ and $C(r) = r^2$. At the closest approach $r = r_0$, where the impact parameter satisfies $u^2 = C(r_0)/A(r_0) = r_0^2/e^{2\Phi(r_0)}$. Hence,

$$\frac{C(r)}{C(r_0)} \frac{A(r_0)}{A(r)} = \frac{r^2}{u^2 e^{2\Phi(r)}}.$$

Substituting these relations into the general deflection-angle expression yields Equation (13).

$$\alpha = 2 \int_{r_0}^{\infty} \frac{(1 - \frac{b(r)}{r})^{-1/2}}{r \sqrt{\frac{r^2}{u^2 e^{2\Phi(r)}} - 1}} dr - \pi, \quad (13)$$

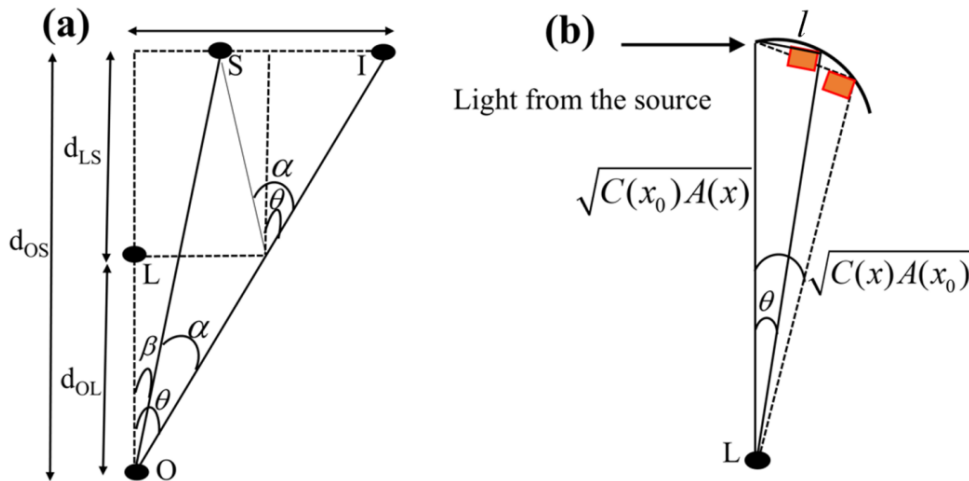


FIGURE 1. (a) The key parameters of gravitational lensing that connect the source S, lens L, observer O, and its image I. (b) Figure shows how the curvature angle θ contributes to the inequality observed in the influence parameter. Adapted with permission from Anuar & Wan Ahmad Tajuddin (2017)

where Equation (13) represents the deflection angle for a general wormhole spacetime, whereas Equation (11) corresponds to the Schwarzschild case. The two expressions have similar forms because the Schwarzschild metric is a special case of the wormhole metric obtained by setting $b(r) = 2Gm$ and $e^{2\Phi(r)} = 1 - 2Gm/r$.

In this scenario, r_0 refers to the maximum curvature radius of the wormhole, with the redshift function term at the throat expressed as $e^{2\Phi(r_0)}$, resulting in $u = r_0/e^{\Phi(r_0)}$. In the weak field approximation and in isotropic coordinates, the metric of a spherically symmetric brane spacetime takes a simplified form (Anuar & Wan Ahmad Tajuddin 2017), as follows

$$ds^2 = -\left(1 + \frac{2\Phi}{c^2}\right)^2 c^2 dt^2 + \left(1 - \frac{2(\Phi - \psi)}{c^2}\right) dX^2, \quad (14)$$

where $\Phi(r)$ represents the Newtonian potential, while $\psi(r)$ serves as the relativistic potential, functioning as a corrections arising from braneworld gravity. Applying the same method, the deflection angle is derived from the above equation, we obtain

$$\alpha = \alpha_R + \frac{1}{c^2} \int_0^s \nabla_{\perp} \psi dl, \quad (15)$$

where α_R is the deflection angle from general relativity alone that represents the original value, with an additional term representing the brane α_{ψ} , $\alpha = \alpha_R + \alpha_{\psi}$. Since the integral term I is $\alpha = I - \pi$ relates to both general relativity and extra dimensions within the context of braneworld gravity. The deflection angle is determined by the interplay of these two elements $\alpha = I_R + I_{\psi} - \pi$. Thus, this shows the correction from extra dimensions, causing slight deviations in light deflection compared to general relativity and suggests further exploration into the extra-dimensional term within the braneworld wormhole lensing model. Hence, by examining the spacetime line element, which accounts for both spherical symmetry and an additional term from the extra dimension in the Braneworld framework (Wong, Harko & Cheng 2011).

$$ds^2 = -M^2 dt^2 + N^2 dr^2 + P^2 (d\theta^2 + \sin^2 \theta d\phi^2) + Q^2 dy^2, \quad (16)$$

where $Y(r)$ denotes the brane embedding function, such that $y = Y(r)$, thus $dy = Y_r dr$, and $ds^2 = -M^2 dt^2 + N^2 dr^2 + P^2 d\theta^2 + P^2 \sin^2 \theta d\phi^2 + Q^2 (Y_r)^2 dr^2$ (Anuar & Wan Ahmad Tajuddin 2017).

We may expand the metrics with rotational terms. The rotational motion of spacetime may contribute to the expansion of wormhole throat radii (Tangphati et al. 2024); it will affect the curvature of the surrounding spacetime. This will consequently affect the gravitational

lens. Hence, it is of particular interest to examine the rotational effect of the braneworld wormhole on its gravitational lensing properties. Substituting the rotational transformation into the metric and simplifying the angular term, we obtain

$$ds^2 = -M^2 dt^2 + N^2 dr^2 + P^2 d\theta^2 + Q^2 (Y_r)^2 dr^2 + P^2 \sin^2 \theta (d\phi - d\phi_{rot})^2, \quad (17)$$

where M denotes the redshift function governing the temporal component; N characterizes the radial metric function; P as the radial term; and $Q(Y_r)$ represents the extra-dimensional contribution. To incorporate rotational effects into the spacetime, we introduce a time-dependent azimuthal shift in the angular coordinate. This is expressed through the relation

$$d\phi_{rot} = \frac{d\phi}{dt} dt = \omega dt, \quad (18)$$

where ω denotes the angular velocity of the rotating wormhole spacetime, defined as $\omega = \partial\phi/\partial t$, and governs the rotational structure of the geometry. This formulation follows the canonical stationary axisymmetric wormhole metric (Teo 1998).

From Equation (18) in (17)

$$ds^2 = -\mu^2 dt^2 + (N^2 + Q^2 (Y_r)^2) dr^2 + P^2 d\theta^2 - P^2 \sin^2 \theta d\phi^2, \quad (19)$$

where

$$\mu^2 = M^2 - P^2 \omega^2 \sin^2 \theta. \quad (20)$$

To relate it to a geometric feature inherent in the wormhole line element, the red shift $M = e^{\Phi(r)}$, the shape function $N^2 + (QY_r)^2 \approx a^2/(1 - (b(r)/r))$, and the radial term $P \approx ar$ introduced. To simplify the induced metric and explicitly incorporate the effect of the extra-dimensional embedding, we introduce an effective radial metric function defined as $W = \sqrt{N^2 + Q^2 Y_r^2}$, thus, $ds^2 = -\mu^2 dt^2 + (N^2 + Q^2 (Y_r)^2) dr^2 + P^2 d\theta^2 - P^2 \sin^2 \theta d\phi^2$. Let $ds^2 = 0$, for light traveling from a distant star along light-like path that is the null geodesic, we may also write $W^2 dr^2 + P^2 \sin^2 \theta d\phi^2 = 0$, hence

$$\left(\frac{d\phi}{dr}\right)^2 = -\frac{W^2}{P^2 \sin^2 \theta}. \quad (21)$$

For the relationship between $d\theta/dt$ is considered by incorporating the redshift function $\mu = e^{\Phi(r,\theta)}$ and the wormhole's radius expansion factor $P \approx ar$, since we consider the light-like condition $ds^2 = 0$, we can write $-\mu^2 dt^2 + P^2 d\theta^2 = 0$, which then we obtain

$$\left(\frac{d\theta}{dt}\right)^2 = \left(\frac{\mu}{P}\right)^2 \Rightarrow \frac{d\theta}{dt} = \frac{\mu}{P} = \frac{e^\Phi}{ar}, \quad (22)$$

where $ar/e^\Phi = u = P_0/\mu_0$, the gravitational lens influence parameter is defined to be greater near the wormhole than in its outer vicinity, which is $u > u_{(\text{region} - \text{away})}$, where $u_{(\text{region} - \text{away})} = P/\mu$. From Figure 2, the relationship between radii near and away from the wormhole simplifies to $l = ((P_0\mu)^2 - (P\mu_0)^2)^{1/2}$. Since $\sin \theta = l/P\mu_0$, $\sin^2 \theta = 1 - P^2/\mu^2\mu_0^2$ and from Equation (21) the rate of change of the wormhole axial angle ϕ with respect to the wormhole radius is

$$\left(\frac{d\phi}{dr}\right)^2 = \frac{W^2}{P^2 \left(\frac{P^2}{u^2 \mu^2} - 1 \right)}. \quad (23)$$

From the shape function mentioned before, the Equation (23) becomes

$$\phi = \int \frac{1}{P} \sqrt{\frac{N^2 + Q^2 Y_r^2}{\frac{P^2}{u^2 \mu^2} - 1}} dr, \quad (24)$$

thus, the axial angle of the wormhole is exhibited in Figure 2(b) (Anuar & Wan Ahmad Tajuddin 2017).

In this study, we extend the classical braneworld wormhole solution to a rotating configuration by incorporating the rotation term ω and a higher-dimensional

correction term $Q^2(Y_r)^2$. The resulting spacetime remains effectively spherically symmetric in the radial direction and retains a Morris-Thorne-like structure. To maintain analytical continuity with previous models and facilitate comparisons, we adopt the same shape function used in the non-rotating case. This choice satisfies the standard wormhole conditions:

- (i) $\lim_{r \rightarrow \infty} \frac{b(r)}{r} = 0$ (asymptotic flatness),
- (ii) $b(r_0) = r_0$ (throat condition),
- (iii) $b'(r_0) < 1$ (flare-out condition).

Despite the additional rotational and brane-induced corrections, the radial component of the metric preserves its fundamental role in defining the wormhole geometry. Therefore, the use of this shape function is valid and consistent within our modified rotating wormhole spacetime. The additive nature of the correction $Q^2(Y_r)^2$ does not violate the essential wormhole structure but rather enriches the geometry with effects from extra-dimensional physics. The divergence of g_{rr} at the throat $r = r_0$ is still preserved, satisfying the minimal surface condition of a traversable wormhole. Figure 3 shows the effect of rotation on gravitational lensing in wormhole geometries. In panel (a), a static (non-rotating) wormhole exhibits strong symmetric lensing, where light rays experience maximal deflection due to the curvature of spacetime. Panels (b) and (c) illustrate how increasing rotational speed alters this behavior. As the rotation becomes more pronounced, the curvature encountered by light rays is modified, leading to asymmetric trajectories and a reduction in the overall deflection angle. In this framework, a ‘low gravitational lens’ does not refer to weaker gravity, but rather to reduced light bending caused by rotational effects near the wormhole throat, such as centrifugal influences.

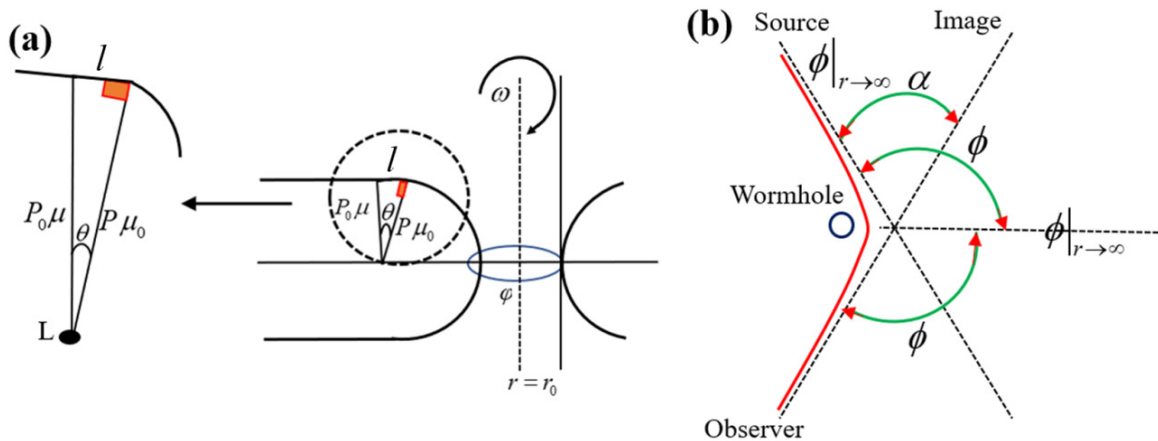


FIGURE 2. (a) Relationship between the curvature angle at the wormhole throat and the impact parameter, inequality representation where $P_0\mu > P\mu_0$, (b) the influence of the wormhole's axial angle on the light deflection angle. Adapted with permission from Anuar & Wan Ahmad Tajuddin (2017)

The deflection angle α can be formulated using the wormhole's axial angle, given by $\alpha = 2\phi - \pi$. From Equation (24), α can be formulated as

$$\alpha = 2 \int_{r_0}^{\infty} \frac{1}{P} \sqrt{\frac{N^2 + Q^2 Y_r^2}{\frac{P^2}{u^2 \mu^2} - 1}} dr - \pi. \quad (25)$$

We may rewrite Equation (25) as the deflection angle $\alpha = I(\varphi) - \pi$, where $I(\varphi)$ represents the axial angle function of the wormhole

$$I(\varphi) = 2 \int_{r_0}^{\infty} \frac{1}{P} \sqrt{\frac{N^2 + Q^2 Y_r^2}{\frac{P^2}{u^2 \mu^2} - 1}} dr. \quad (26)$$

Thus, after factorizing and expanding the term $(N^2 + Q^2 Y_r^2)^{1/2}$ by binomial series, the Equation (25) can be arranged to showcase both the classical (free brane) contributions and the influence of brane (brany) effects.

$$\alpha = 2 \int_{r_0}^{\infty} \frac{N}{P \sqrt{P^2 / (u^2 \mu^2) - 1}} dr + 2 \int_{r_0}^{\infty} \frac{1}{P \sqrt{P^2 / (u^2 \mu^2) - 1}} \left(\frac{1}{2} \frac{(QY_r)^2}{N} - \frac{1}{8} \frac{(QY_r)^4}{N^3} + \dots \right) dr - \pi. \quad (27)$$

This shows that the axial function of wormhole includes two parts, capturing both classical and braneworld contributions, $I(\varphi) = I(\varphi)_{\text{classic}} + I(\varphi)_{\text{brany}}$, respectively. Accordingly,

$$I(\varphi)_{\text{class}} = 2 \int_{r_0}^{\infty} \frac{N}{P \sqrt{P^2 / (u^2 \mu^2) - 1}} dr, \quad (28)$$

$$I(\varphi)_{\text{brany}} = 2 \int_{r_0}^{\infty} \frac{1}{P \sqrt{P^2 / (u^2 \mu^2) - 1}} \left(\frac{1}{2} \frac{(QY_r)^2}{N} - \frac{1}{8} \frac{(QY_r)^4}{N^3} + \dots \right) dr. \quad (29)$$

Here, we define $N^2 = (1 - (b(r)/r))^{-1}$ to represent the purely general relativity of wormhole, and $(QY_r)^2 = (a^2 - 1)(1 - (b(r)/r))^{-1}$, for the braneworld wormhole extra-dimensional term. Hence, the complete integral form of the deflection angle due to a braneworld wormhole is given by

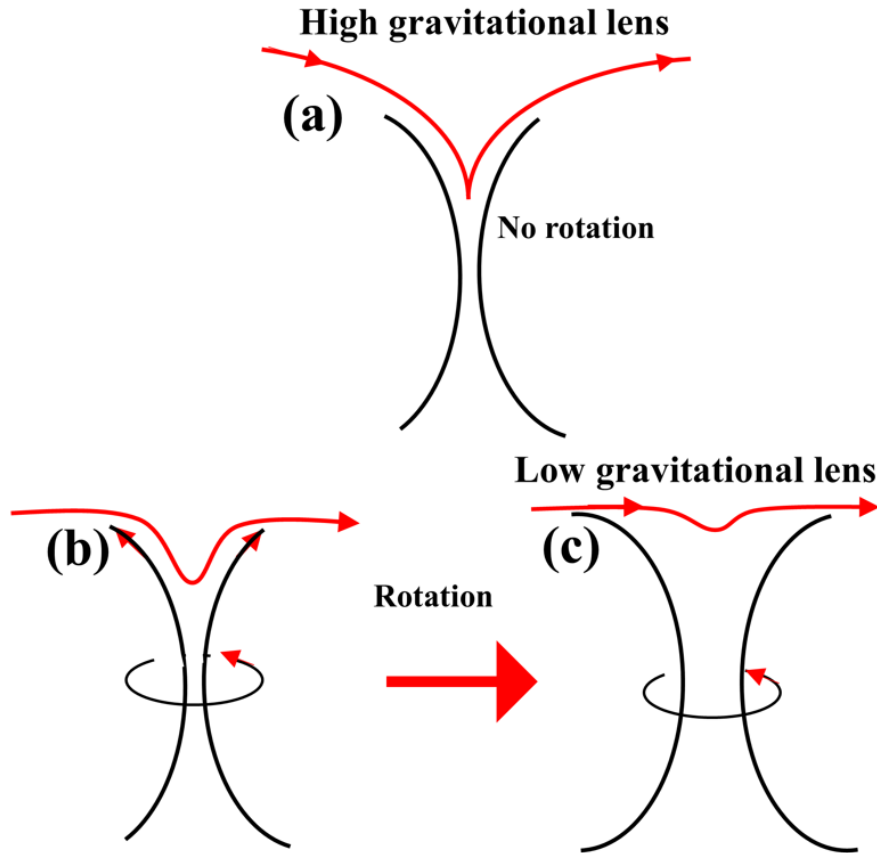


FIGURE 3. (a) The static wormhole produces strong symmetric lensing with maximum deflection; (b) moderate rotation introduces asymmetry in light bending and slightly reduces the lensing effect; and (c) high rotation leads to minimal deflection as rotational effects dominate the curvature experienced by light rays

$$\alpha = 2 \int_{r_0}^{\infty} \frac{(1 - \frac{b(r)}{r})^{-1/2}}{ar \sqrt{\frac{a^2 r^2}{u^2 (e^{2\Phi(r)} - p^2 \omega^2 \sin^2 \theta)} - 1}} dr + 2 \int_{r_0}^{\infty} \frac{1}{ar \sqrt{\frac{a^2 r^2}{u^2 (e^{2\Phi(r)} - p^2 \omega^2 \sin^2 \theta)} - 1}} dr - \pi. \quad (30)$$

$$\left(\frac{1}{2} \frac{(a^2 - 1)}{(1 - \frac{b(r)}{r})^{-1/2}} - \frac{1}{8} \frac{(a^2 - 1)^2}{(1 - \frac{b(r)}{r})^{-3/2}} + \dots \right) dr - \pi.$$

Let $\beta = a^2 - 1$, we have $\lambda = \frac{\beta}{2} - \frac{\beta^2}{8} + \frac{\beta^3}{16} - \frac{5\beta^4}{128} + \frac{7\beta^5}{256} \dots \pm O(\beta)$, and the Equation (30) can be given as the braneworld wormhole deflection angle,

$$\alpha_{\text{brane}} = 2(1 + \lambda) \int_{r_0}^{\infty} \frac{(1 - \frac{b(r)}{r})^{-1/2}}{ar \sqrt{\frac{a^2 r^2}{u^2 (e^{2\Phi(r)} - p^2 \omega^2 \sin^2 \theta)} - 1}} dr - \pi. \quad (31)$$

If the spacetime expansion rate or evolution factor approaches $a \rightarrow 1$, $\lambda \rightarrow 0$, the expression reverts to the classical deflection angle when considering a purely general relativity framework without any extra-dimensional contributions. In this formulation, the parameter a is introduced as an effective spacetime evolution factor following Anuar and Wan Ahmad Tajuddin (2017), where its deviation from unity parametrizes the influence of extra-dimensional effects on the wormhole geometry and, consequently, on the gravitational lensing. This indicates that the cosmological model implies the existence of extra dimensions in the braneworld scenario. Near the wormhole, the spacetime expansion rate $a = a(t)$ may slightly exceed unity, which can be related to the Hubble constant $H = \dot{a}(t)/a(t)$, leading to $a(t) = e^{Ht}$, we obtain $a^2(t) = e^{2Ht}$. Thus, with a non-zero Hubble constant as the universe dynamic, now $\beta = e^{2Ht} - 1$ is the expansion parameter and λ is the corresponding series-expansion correction term, defined as

$$\lambda = \frac{e^{2Ht} - 1}{2} - \frac{(e^{2Ht} - 1)^2}{8} + \frac{(e^{2Ht} - 1)^3}{16} + \dots. \quad (32)$$

This implies that a nonzero Hubble constant signifies a dynamic universe and, by itself, indicates a braneworld cosmology. Equation (31) can be expanded by applying the wormhole boundary conditions for both its interior and exterior regions, which include the classical general relativity term along with the brane extra term $\alpha_{\text{brane}} = \alpha_{GR} + \alpha_{\text{braneExtra}}$, as

$$\alpha_{GR} = 2 \left(\int_{r_0}^{r_s} \frac{1}{ar \sqrt{\frac{a^2 r^2}{u^2 (e^{2\Phi(r)} - p^2 \omega^2 \sin^2 \theta)} - 1}} \cdot \frac{1}{\sqrt{1 - \frac{b(r)}{r}}} dr + \int_{r_s}^{\infty} \frac{1}{ar \sqrt{\frac{a^2 r^2}{u^2 (e^{2\Phi(r)} - p^2 \omega^2 \sin^2 \theta)} - 1}} \cdot \frac{1}{\sqrt{1 - \frac{b(r)}{r}}} dr \right) - \pi, \quad (33)$$

and the brane extra term is

$$\alpha_{\text{braneExtra}} = 2\lambda \left(\int_{r_0}^{r_s} \frac{1}{ar \sqrt{\frac{a^2 r^2}{u^2 (e^{2\Phi(r)} - p^2 \omega^2 \sin^2 \theta)} - 1}} \cdot \frac{1}{\sqrt{1 - \frac{b(r)}{r}}} dr + \int_{r_s}^{\infty} \frac{1}{ar \sqrt{\frac{a^2 r^2}{u^2 (e^{2\Phi(r)} - p^2 \omega^2 \sin^2 \theta)} - 1}} \cdot \frac{1}{\sqrt{1 - \frac{b(r)}{r}}} dr \right). \quad (34)$$

The region $r_0 < r < r_s$ defines the interior of the wormhole, lying between the wormhole's Schwarzschild radius and throat. In this region, the shape function is given by $b(r) = B(1 - e^{-r/r_s - r})$ where $B = \frac{r_0}{1 - e^{-r_0/r_s - r_0}}$.

Following the approach of Mishra and Chakraborty (2018), we adopt the generalized redshift treatment, with the correction term specified by Anuar and Wan Ahmad Tajuddin (2017), the red shift function can be written as

$$\Phi(r) = \frac{1}{2} \ln \left[1 - \frac{b(r)}{r} + \frac{\varepsilon}{r^2} \right], \quad (35)$$

where $\varepsilon > 0$ is a regularization parameter with dimensions of length squared. The term ε/r^2 ensures $e^{2\Phi(r)}$ that remains positive at the wormhole throat, where $b(r_0) = r_0$ thereby avoiding the formation of an event horizon.

At the throat,

$$e^{2\Phi(r_0)} = 1 - \frac{b(r_0)}{r_0} + \frac{\varepsilon}{r_0^2} = \frac{\varepsilon}{r_0^2} > 0.$$

For numerical calculation convenience, we assume an ideal case (minimal or no correction), whereby the redshift correction parameter is fixed at $\varepsilon = 1$. Thus from Equation (35) we obtain

$$\Phi(r) = \frac{1}{2} \ln \left[1 - \frac{r_0}{r} \left(\frac{1 - e^{-r/r_s - r}}{1 - e^{-r_0/r_s - r_0}} \right) + \frac{1}{r^2} \right]. \quad (36)$$

Here, in the exterior region, defined by the radial interval $r_s < r < \infty$, extends from just beyond the wormhole's Schwarzschild radius to asymptotically flat spacetime. This region corresponds to where the shape function assumes a constant value, given by

$$b(r) = b(r_s) = B = \frac{r_0}{1 - e^{-r_0/r_s - r_0}}$$

and the red shift function is expressed as

$$\Phi(r) = \frac{1}{2} \ln \left(1 - \frac{B}{r} \right). \quad (37)$$

Figure 4 illustrates the change of the deflection angle with respect to the expansion factor, where $a=1.06$ to 1.6, assuming a rotating wormhole geometry with throat radius (m) $r_o = 200$, Schwarzschild radius (m) $r_s = 10000$, redshift function error term as $\varepsilon = 1$, and rotation parameter $\omega = 0.005$ at an angle $\theta = 90^\circ$. Since

$b(r_o) = r_o$, the impact parameter simplifies to $u = r_o^2$, where the ratio between these radii must conform to the geometric characteristics of a wormhole, which represents a spacetime that is neither as extremely curved as that of a black hole nor as weakly curved as that of a standard Schwarzschild object. The classical GR deflection angle α_{GR} decreases, while the brane-induced rotational correction $\alpha_{braneExtra}$ increases with expansion, demonstrating the growing influence of rotation-enhanced extra-dimensional effects (Table 1).

TABLE 1. Deflection angles in degrees as a function of the increase of the space expansion rate

a	α_{GR}	$\alpha_{braneExtra}$	α_{brane}
1.06	47.802	-0.08497	47.717
1.07	47.326	0.39115	47.717
1.08	46.859	0.85845	47.717
1.09	46.4	1.3172	47.717
1.1	45.95	1.7676	47.717
1.11	45.507	2.2099	47.717
1.12	45.073	2.6444	47.717
1.13	44.646	3.0712	47.718
1.14	44.227	3.4906	47.718
1.15	43.815	3.9028	47.718
1.16	43.41	4.308	47.718
1.17	43.013	4.7065	47.719
1.18	42.621	5.0984	47.72
1.19	42.237	5.484	47.721
1.2	41.859	5.8634	47.722
1.21	41.487	6.2371	47.724
1.22	41.121	6.6051	47.726
1.23	40.761	6.9677	47.729
1.24	40.407	7.3253	47.732
1.25	40.059	7.678	47.737
1.26	39.716	8.0262	47.742
1.27	39.378	8.3701	47.748
1.28	39.046	8.7101	47.756
1.29	38.719	9.0466	47.766
1.3	38.397	9.3799	47.777
1.31	38.08	9.7104	47.79
1.32	37.768	10.039	47.806
1.33	37.46	10.365	47.825
1.34	37.157	10.69	47.847
1.35	36.859	11.014	47.872
1.36	36.565	11.337	47.902
1.37	36.275	11.661	47.936

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1.38	35.989	11.986	47.976
1.39	35.708	12.313	48.021
1.4	35.43	12.642	48.073
1.41	35.157	12.975	48.132
1.42	34.887	13.312	48.199
1.43	34.621	13.654	48.275
1.44	34.359	14.002	48.361
1.45	34.1	14.358	48.458
1.46	33.845	14.722	48.567
1.47	33.593	15.097	48.69
1.48	33.345	15.482	48.827
1.49	33.1	15.88	48.98
1.5	32.859	16.293	49.151
1.51	32.62	16.721	49.341
1.52	32.385	17.168	49.553
1.53	32.153	17.634	49.787
1.54	31.924	18.122	50.045
1.55	31.697	18.634	50.331
1.56	31.474	19.173	50.647
1.57	31.254	19.74	50.993
1.58	31.036	20.339	51.375
1.59	30.821	20.972	51.793
1.6	30.609	21.643	52.252

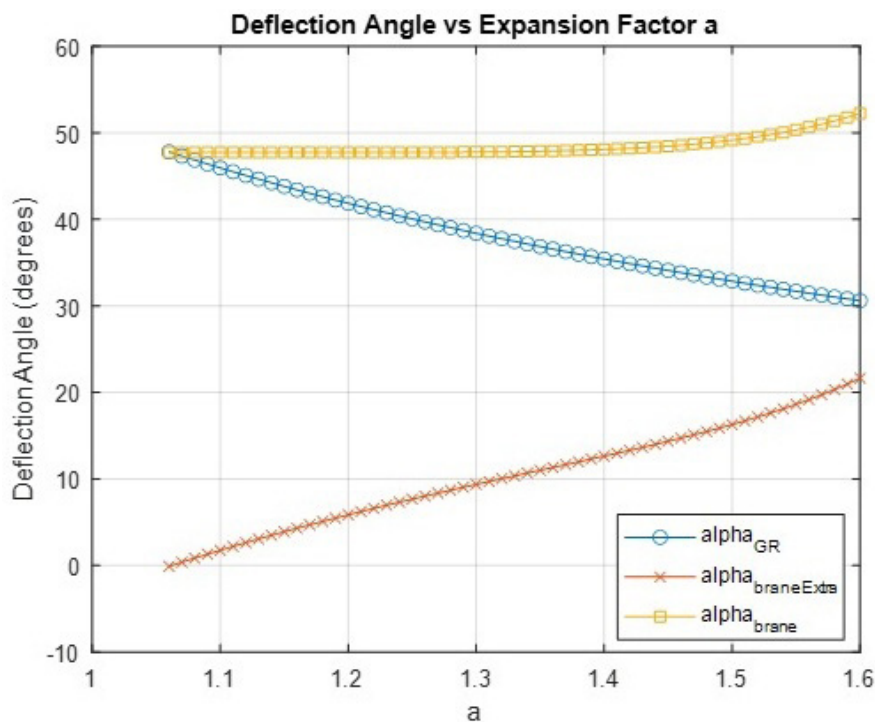


FIGURE 4. Deflection angle behavior influenced by the expansion factor a for a rotating wormhole in the braneworld scenario

CONCLUSION

Our analysis demonstrates that in a braneworld scenario, the gravitational lensing signature of rotating wormholes significantly deviates from those predicted by standard general relativity. The rotation, characterized by ω , is further modified by the presence of extra-dimensional terms arising from bulk-brane interactions. These corrections appear explicitly in the deflection angle expression, where ω -dependent terms are amplified and influence photon trajectories. The resulting redshift function captures both rotational and extra-dimensional contributions to time dilation. Additionally, the shape function is encoded within the integral form of the deflection angle, sensitive to the background spacetime curvature and embedding geometry. The metric deformation due to the braneworld structure yields measurable differences in the bending of light, particularly at higher expansion rates, linking lensing behavior to cosmic evolution. These results indicate that extra-dimensional effects may produce distinctive modifications to gravitational lensing signatures, motivating future studies to assess their potential observational detectability. In summary, the study offers a novel lensing framework that not only generalizes rotating wormhole models but also provides a foundation for identifying braneworld signatures in future astrophysical observation.

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