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Degree Square Subtraction Energy of Non-Commuting Graph for Dihedral Groups

(Tenaga Tolak Darjah Kuasa Dua bagi Graf Tak Kalis Tukar Tertib untuk Kumpulan Dwihedron)

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ABSTRACT

The non-commuting graph on a finite G, denoted by Γ_G, with the set of non-central elements of G as the vertex set and two distinct vertices are adjacent whenever they do not commute in G. In this paper, we discuss the spectrum, spectral radius and degree square subtraction energy of Γ_G for dihedral groups of order $2n$, D_{2n}, where $n\geq3$. It is found that the obtained energy here is equal to twice its spectral radius and there is a relationship with the degree subtraction energy that was described in previous literature.

Keywords: Degree square subtraction matrix; dihedral group; non-commuting graph; the energy of a graph

ABSTRAK

Graf tak kalis tukar tertib ditakrifkan pada suatu kumpulan terhingga G, ditandakan dengan Γ_G, dengan set unsur bukan pusat G sebagai set bucu dan dua bucu berbeza adalah bersebelahan apabila mereka tak kalis tukar tertib dalam G. Dalam makalah ini, kita membincangkan spektrum, jejari spektrum dan tenaga tolak darjah kuasa dua bagi Γ_G untuk kumpulan dwihedron peringkat $2n$, D_{2n}, yang $n\geq3$. Didapati bahawa tenaga yang diperoleh ini adalah sama dengan dua kali jejari spektrumnya dan terdapat hubungan dengan tenaga tolak darjah yang telah diterangkan dalam kajian terdahulu.

Kata kunci: Graf tak kalis tukar tertib; kumpulan dwihedron; matriks tolak darjah kuasa dua; tenaga graf

INTRODUCTION

Graph energy and its variants were originally developed to investigate mathematical problems, but have since found applications in several fields of science and engineering, some of which are surprising and mysterious (Gutman & Furtula 2019). The applications can be found in Chemistry, for instance, crystallography (Yuge 2018), the theory of macromolecules (Dhanalakshmi, Rao & Sivakumar 2015), and the analysis and comparison of protein sequences (Sun, Xu & Zhang 2016). The other applications are in network analysis (Huang et al. 2019), air transportation (Jiang et al. 2016), and satellite communication (Akram & Naz 2018). Moreover, related applications to computer science (Praba, Deepa & Chandrsekaran 2016) and process analysis (Musulin 2014) also have been reported, in line with engineering complex systems design and analysis (Sinha & Suh 2018), and the construction of spacecraft (Pugliese & Nilchiani 2017). In other fields, the application is also found in pattern and face recognition (Angadi & Hatture 2019), object identification (Xiao, Song & Hall 2011), image analysis (Zhang et al. 2013), and processing for classifying high-resolution satellite images (Ankayarkanni & Leni 2014). In addition, there are applications in health and medicine (Singh, Baths & Kumar 2014), epidemics (Van Mieghem & van de Bovenkamp 2015), neuronal networks (Dasgupta et al. 2015), and Alzheimer's disease (Daianu et al. 2015).

Most research on graphs defined on groups primarily focuses on computing various parameters of graph theory. By using graphs, we may discover new information about groups, and be able to identify classes of groups that are interesting by putting conditions on various graphs defined for the groups. We may also discover the automorphism group in the process of finding graphs of finite groups properties (Cameron 2023).

The best example of graphs defined on group is the non-commuting graph on G, denoted by Γ_G, with the set of non-central elements of G as the vertex set of Γ_G. Two vertices $v_p \neq v_q$ are adjacent whenever $v_p v_q \neq v_q v_p$ (Abdollahi, Akbari & Maimani 2006). The $n \times n$ adjacency matrix of Γ_G is denoted by $A(\Gamma_G) = [a_{pq}]$, in which $a_{pq} = 1$, for adjacent v_p and v_q, and otherwise, $a_{pq} = 0$. For an $n \times n$ identity matrix I_n, the characteristic polynomial of $A(\Gamma_G)$ is $P_{A(\Gamma G)}(\lambda) = |\lambda I_n - A(\Gamma_G)|$. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of Γ_G as the roots of $P_{A(\Gamma G)}(\lambda) = 0$. The list $\lambda_1, \lambda_2, \ldots, \lambda_n$ with their respective multiplicities $k_1, k_2, \ldots, k_n$ is the spectrum of Γ_G, denoted by $\mathrm{Spec}(\Gamma_G) = \{\lambda_1^{k_1}, \lambda_2^{k_2}, \ldots, \lambda_n^{k_n}\}$. Moreover, the spectral radius of Γ_G is defined as $\rho(\Gamma_G) = max\ \{|\lambda| : \lambda \in Spec(\Gamma_G)\}$ (Horn & Johnson 1985). Several works focus on the spectral radius of Γ_G with regard to the degree sum matrix (Romdhini & Nawawi 2022a) and neighbors degree sum matrix (Romdhini, Nawawi & Chen 2023).

Moreover, Gutman was the first to pioneer the adjacency energy of a finite graph in 1978, it is defined as $E_{A(\Gamma G)} = \sum_{i=1}^{n} |\lambda_i|$. If $E_{A(\Gamma G)}$ more than $E_A(K_n)$ or $E(\Gamma_G) > 2(n-1)$, then Γ_G can be classified as hyperenergetic (Li, Shi & Gutman 2012). It should be noted as well that the energy value is neither an odd number (Bapat & Pati 2004) nor the square root of an odd number (Pirzada & Gutman 2008).

Macha and Shinde introduced a new graph matrix in 2022, the degree square subtraction (DSS) matrix of Γ_G, $DSS(\Gamma_G)$. Let d_{v_i} be the degree of v_i, then $DSS(\Gamma_G) = [dss_{pq}]$ in which (p,q) -th entry is

$$dss_{pq} = \begin{cases} d_{v_p}^2 - d_{v_q}^2, & \text{if } v_p \neq v_q \\ 0, & \text{if } v_p = v_q. \end{cases}$$

In this note, we work on the non-abelian dihedral group of order $2n$, $n \geq 3$, denoted by $D_{2n} = \langle a, b : a^n = b^2 = e, bab = a^{-1} \rangle$ (Aschbacher 2000). The centre of D_{2n},

$$Z(D_{2n}) = \begin{cases} \{e\}, & \text{for odd } n \\ \left\{e, a^{\frac{n}{2}}\right\}, & \text{for even } n. \end{cases}$$

The centralizer of a^i in D_{2n} is $C_{D_{2n}}(a^i) = \{a^j : 1 \leq j \leq n\}$ and for $a^i b$ is

$$C_{D_{2n}}(a^i b) = \begin{cases} \{e, a^i b\}, & \text{for odd } n \\ \left\{e, a^{\frac{n}{2}}, a^i b, a^{\frac{n}{2}+i} b\right\}, & \text{for even } n. \end{cases}$$

Recent research on the graph energy of Γ_G for D_{2n} has been published in Romdhini, Nawawi and Chen (2022) and Romdhini and Nawawi (2022b). They discussed degree exponent sum and maximum and minimum degree matrices. Moreover, the graph energy of Γ_G for D_{2n} was discussed by Romdhini and Nawawi (2023) corresponds with the degree subtraction (*DSt*) matrix. Let $E_{DSt}(\Gamma_G)$ be the *DSt*-energy of Γ_G for D_{2n}, then

$$E_{DSt}(\Gamma_G) = \begin{cases} 2(n-2)\sqrt{n(n-1)}, & \text{for odd } n \\ 2(n-4)\sqrt{n(n-2)}, & \text{for even } n. \end{cases} \quad (1)$$

Inspired by this, the authors apply the degree square subtraction matrix of Γ_G for D_{2n} and explore the characteristic polynomial, spectrum, and energy. In the end, the relationship between those energies based on the *DSS* and *DSt*-matrices is presented.

The methodology starts with constructing the degree square subtraction matrix of Γ_G, determining the spectrum of Γ_G, investigating $\rho(\Gamma_G)$, calculating the degree square subtraction energy, then studying the relationship between $\rho(\Gamma_G)$ and the degree square subtraction energy of Γ_G. Finally, the observation of the hyperenergetic property is presented.

PRELIMINARIES

Let $G_1 = \{a^i : 1 \leq i \leq n\} \backslash Z(D_{2n})$ and $G_2 = \{a^i b : 1 \leq i \leq n\}$, where $G_1 \cup G_2$ is subset of D_{2n}. We investigate Γ_G, where G is either G_1, G_2 or $G_1 \cup G_2$. Morever, the formula of degree square subtraction energy of Γ_G is

$$E_{DSS}(\Gamma_G) = \sum_{i=1}^{n} |\lambda_i|, \quad (2)$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $DSS(\Gamma_G)$. The *DSS*- spectral radius of Γ_G is

$$\rho_{DSS}(\Gamma_G) = max\{|\lambda|: \lambda \in Spec(\Gamma_G)\}. \quad (3)$$

For odd n, Γ_G has $2n$ - 1 vertices, while there are $2n$ - 2 vertices for even n. Then, Γ_G corresponds to the degree square subtraction matrix can be stated as a hyperenergetic if the degree square subtraction energy holds:

$$E_{DSS}(\Gamma_G) > \begin{cases} 4(n-1), & \text{for odd } n \\ 4(n-1)-2, & \text{for even } n. \end{cases} \quad (4)$$

We recall some previous results related to the vertex degree and isomorphism of graphs for constructing the -matrix.

Theorem 2.1 (Khasraw, Ali & Haji 2020) In Γ_G for $G = G_1 \cup G_2$, the degree of

(1) a^i on Γ_G is $d_{a^i} = n$, and

(2) a^ib on Γ_G is $d_{a^ib} = \begin{cases} 2(n-1), & \text{if } n \text{ is odd} \\ 2(n-2), & \text{if } n \text{ is even.} \end{cases}$

Theorem 2.2 (Khasraw, Ali & Haji 2020) In Γ_G,

(1) if $G = G_1$, then $\Gamma_G \cong \overline{K}_m$, where $m = |G_1|$,

(2) if $G = G_2$, then $\Gamma_G \cong \begin{cases} K_n, & \text{for odd } n \\ K_n - \frac{n}{2}K_2, & \text{for even } n, \end{cases}$

where K_n is a complete graph on n vertices, $\overline{K}_n$ is its complement, and $\frac{n}{2}$ K_2 is $\frac{n}{2}$ copies of K_2.

From Theorems 2.1 and 2.2, we are able to construct the degree square subtraction matrix of Γ_G. We first present a result from Ramane and Shinde (2017) for formulating the characteristic polynomial of Γ_G.

Lemma 2.1 (Ramane & Shinde 2017) For real numbers *a,b,c* and *d*, and an $n \times n$ matrix J_n in which all entries are 1, then

$$\begin{vmatrix} (\lambda+a)I_{n_1} - aJ_{n_1} & -cJ_{n_1\times n_2} \\ -dJ_{n_2\times n_1} & (\lambda+b)I_{n_2} - bJ_{n_2} \end{vmatrix}_{(n_1+n_2)\times(n_1+n_2)}$$

can be simplified as

$$(\lambda+a)^{n_1-1}(\lambda+b)^{n_2-1}\big((\lambda-(n_1-1)a)$$
$$(\lambda-(n_2-1)b) - n_1n_2cd\big),$$

where $1 \leq n\_1$, n_2 $\leq n$ and $n_1 + n_2 = n$.

MAIN RESULTS

Let us start with the degree square subtraction energy of Γ_G for both $G = G_1$ and $G = G_2$.

Theorem 3.1 In Γ_G for $G = D_{2n}$ and $E_{DSS}(\Gamma_G)$ be the degree square subtraction energy of Γ_G. If $G = G_1$ or $G = G_2$, then $E_{DSS}(\Gamma_G) = 0$.

Proof.

1. From Theorem 2.2 (1), where $G = G_1$, we have $\Gamma_G \cong \overline{K}_m$ which means every vertex has a degree zero in Γ_G. For the first case when n is odd, we know that $m = |G_1| = n$-1. By removing e and $a^{\frac{n}{2}}$ for even n, we have $m = n$-2. So the degree square subtraction matrix of Γ_G for odd n is $DSS(\Gamma_G) = [0]_{n-1}$, and for even n, $DSS(\Gamma_G) = [0]_{n-2}$. It is clear that 0 is the only eigenvalue of $DSS(\Gamma_G)$. Hence, according to the formula on Equation (2), $E_{DSS}(\Gamma_G) = 0$.

2. Theorem 2.2 (2)) implies $\Gamma_G \cong K_n$ for $G = G_2$ and n is odd. Consequently, for every vertex in Γ_G, the degree is n-1. Therefore, for $v_p \neq v_q$, the (p,q) -th entry of $DSS(\Gamma_G)$ is $(n-1)^2 - (n-1)^2 = 0$ and otherwise, it is zero. It is clear that $DSS(\Gamma_G)$ is a zero matrix. Therefore $E_S(\Gamma_G) = 0$. Meanwhile, the observation of even n from Theorem 2.2 (2), we have $\Gamma_G \cong K_n - \frac{n}{2} K_2$, which means $d_{a^ib} = n$-2. By the definition of $DSS(\Gamma_G)$, for $v_p \neq v_q$, the (p,q)-th entry is $(n-2)^2 - (n-2)^2 = 0$ and otherwise, it is zero. Consequently, $DSS(\Gamma_G)$ is a zero matrix of size $n \times n$. Similarly, $E_{DSS}(\Gamma_G) = 0$.

The characteristic polynomial of $DSS(\Gamma_G)$, $\rho_{DSS}(\Gamma_G)(\lambda)$, and the degree square subtraction energy of Γ_G for $G = G_1 \cup G_2$ are presented below:

Theorem 3.2 In Γ_G for $G = G_1 \cup G_2 \subset D_{2n}$, then

(1) for odd n, $P_{DSS(\Gamma_G)}(\lambda) = (\lambda)^{2n-3}(\lambda^2 + n(n-1)$
$(3n^2 - 8n + 4)^2)$,

(2) for even n, $P_{DSS(\Gamma_G)}(\lambda) = (\lambda)^{2n-4}(\lambda^2 + n(n-2)$
$(3n^2 - 16n + 16)^2)$.

Proof.

Let n is odd, from the fact that $Z(D_{2n}) = \{e\}$, then Γ_G has $2n$ - 1 vertices, where $G = G_1 \cup G_2$. Now let $G_1 = \{a, a^2, \ldots, a^{n-1}\}$ and $G_2 = \{b, ab, a^2 b, \ldots, a^{n-1} b\}$. Using the centralizer of a^i in D_{2n} which is $\{e, a, a^2, \ldots, a^{n-1}\}$, then a^i, for $1 \leq i \leq n - 1$, is not adjacent to all other vertices of G_1, but adjacent to all vertices of G_2. Meanwhile, for $1 \leq i \leq n$, $C_{D_{2n}}(a^ib) = \{e, a^ib\}$ implies the vertex a^ib is always adjacent to all other members of $G_1 \cup G_2$. According to

Theorem 2.1 for $1 \leq i \leq n$, we have $d_{a^i} = n$ and $d_{a^ib} = 2(n - 1)$. A $(2n - 1) \times (2n - 1)$ degree square subtraction matrix for Γ_G is

$$DSS(\Gamma_G) = \begin{array}{c} \\ a \\ a^2 \\ \vdots \\ a^{n-1} \\ b \\ ab \\ \vdots \\ a^{n-1}b \end{array} \begin{array}{c} \begin{array}{cccc:cccc} a & a^2 & \cdots & a^{n-1} & b & ab & \cdots & a^{n-1}b \end{array} \\ \left[\begin{array}{cccc:cccc} 0 & 0 & \cdots & 0 & -3n^2+8n-4 & -3n^2+8n-4 & \cdots & -3n^2+8n-4 \\ 0 & 0 & \cdots & 0 & -3n^2+8n-4 & -3n^2+8n-4 & \cdots & -3n^2+8n-4 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & -3n^2+8n-4 & -3n^2+8n-4 & \cdots & -3n^2+8n-4 \\ \hdashline 3n^2-8n+4 & 3n^2-8n+4 & \cdots & 3n^2-8n+4 & 0 & 0 & \cdots & 0 \\ 3n^2-8n+4 & 3n^2-8n+4 & \cdots & 3n^2-8n+4 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 3n^2-8n+4 & 3n^2-8n+4 & \cdots & 3n^2-8n+4 & 0 & 0 & \cdots & 0 \end{array}\right] \end{array}.$$

Then, the degree square subtraction matrix of Γ_G can be expressed as

$$DSS(\Gamma_G) = \begin{bmatrix} 0_{n-1} & -3n^2 + 8n - 4)J_{(n-1)\times n} \\ (3n^2 - 8n + 4)J_{n\times(n-1)} & 0_n \end{bmatrix},$$

and the determinant herewith is the characteristic polynomial for $DSS\ (\Gamma_G)$,

$$P_{DSS(\Gamma_G)}(\lambda) = \begin{vmatrix} \lambda I_{n-1} & (3n^2 - 8n + 4)J_{(n-1)\times n} \\ (-3n^2 + 8n - 4)J_{n\times(n-1)} & \lambda I_n \end{vmatrix}.$$

Based on Lemma 2.1, with $w = x = 0$, $y = 3n^2 - 8n + 4$, $z = - 3n^2 + 8n - 4$, $n_1 = n - 1$, and $n_2 = n$, therefore

$$P_{DSS(\Gamma_G)}(\lambda) = (\lambda)^{2n-3}(\lambda^2 + n(n-1)(3n^2 - 8n + 4)^2).$$

2. Let n is even and $G = G_1 \cup G$. As we know that $Z(D_{2n}) = \{e, a^{\frac{n}{2}}\}$, so Γ_G has $2n - 2$ vertices, and this actually $n - 2$ vertices from a^i, for $1 \leq i \neq \frac{n}{2} < n$, and n vertices of a^ib, where $1 \leq i \leq n$. We denote G_1 as $\{a, a^2, \ldots, a^{\frac{n}{2}-1}, a^{\frac{n}{2}+1}, \ldots, a^{n-1}$ and $G_2 = \{b, ab, a^2 b, \ldots, a^{n-1}b\}$. Using the centralizer of a^i in D_{2n}, then every vertex of G_1 is adjacent to all vertices of G_2. Since $C_{D_{2n}}(a^ib) = \{e, a^{\frac{n}{2}}, a^ib, a^{\frac{n}{2}+i}b\}$, then a^ib and $a^{\frac{n}{2}+i}b$ are not adjacent in Γ_G. By Theorem 2.1, then $d_{a^i} = n$ and $d_{a^ib} = 2(n - 2)$, which means $DSS\ (\Gamma_G)$ is $(2n - 2) \times (2n - 2)$ matrix as follows

$$\begin{array}{c} \\ a \\ \vdots \\ a^{\frac{n}{2}-1} \\ a^{\frac{n}{2}+1} \\ \vdots \\ a^{n-1} \\ b \\ \vdots \\ a^{n-1}b \end{array} \begin{array}{c} \begin{array}{cccccc:ccc} a & \cdots & a^{\frac{n}{2}-1} & a^{\frac{n}{2}+1} & \cdots & a^{n-1} & b & \cdots & a^{n-1}b \end{array} \\ \left[\begin{array}{cccccc:ccc} 0 & \cdots & 0 & 0 & \cdots & 0 & -3n^2+16n-16 & \cdots & -3n^2+16n-16 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & -3n^2+16n-16 & \cdots & -3n^2+16n-16 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & -3n^2+16n-16 & \cdots & -3n^2+16n-16 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & -3n^2+16n-16 & \cdots & -3n^2+16n-16 \\ \hdashline 3n^2-16n+16 & \cdots & 3n^2-16n+16 & 3n^2-16n+16 & \cdots & 3n^2-16n+16 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 3n^2-16n+16 & \cdots & 3n^2-16n+16 & 3n^2-16n+16 & \cdots & 3n^2-16n+16 & 0 & \cdots & 0 \end{array}\right] \end{array}.$$

The DSS-matrix of Γ_G can be written as

$$DSS(\Gamma_G) = \begin{bmatrix} 0_{n-1} & (-3n^2 + 8n - 4)J_{(n-1)\times n} \\ (3n^2 - 8n + 4)J_{n\times(n-1)} & 0_n \end{bmatrix},$$

and $\rho_{DSS}(\Gamma_G)\ (\lambda)$ is the following determinant:

$$P_{DSS(\Gamma_G)}(\lambda) = \begin{vmatrix} \lambda I_{n-1} & (3n^2 - 8n + 4)J_{(n-1)\times n} \\ (-3n^2 + 8n - 4)J_{n\times(n-1)} & \lambda I_n \end{vmatrix}.$$

Repeated application of Lemma 2.1, with $a = b = 0$, $c = 3n^2 - 16n + 16$, $d = -3n^2 + 16n - 16$, $n_1 = n - 2$, and $n_2 = n$, we get

$$P_{DSS(\Gamma_G)}(\lambda) = (\lambda)^{2n-3}(\lambda^2 + n(n-1)(3n^2 - 8n + 4)^2).$$

In the next results, we prove the DSS-spectral radius, $\rho_{DSS}(\Gamma_G)$, and DSS-energy of Γ_G for $G = G_1 \cup G_2$.

Theorem 3.3 In Γ_G for $G = G_1 \cup G_2 \subset D_{2n}$,

(1) $\rho_{DSS}(\Gamma_G) = (3n-2)(n-2)\sqrt{n(n-1)}$, for odd n, and

(2) $\rho_{DSS}(\Gamma_G) = (3n-4)(n-4)\sqrt{n(n-2)}$, for even n.

Proof.

1. Based on Theorem 3.2 (1) when n is odd, we have $P_{DSS}(\Gamma_G)\ (\lambda)$ which implies three eigenvalues of Γ_G. Then, we get $\lambda_1 = 0$ of multiplicity $2n - 3$. The other two eigenvalues are $\lambda_1, 3 = \pm\, i(3n - 2)(n - 2)\sqrt{n(n-1)}$ as roots of the quadratic polynomial. Thus, the DSS-spectrum of Γ_G is

$$Spec(\Gamma_G) = \left\{\left(i(3n-2)(n-2)\sqrt{n(n-1)}\right)^1, (0)^{2n-3}, \left(-i(3n-2)(n-2)\sqrt{n(n-1)}\right)^1\right\}.$$

Now for $i = 1,2,3$, based on the formula on Equation (3), the maximum of absolute eigenvalues $|\lambda_i|$ is the DSS-spectral radius of Γ_G,

$$\rho_{DSS}(\Gamma_G) = (3n-2)(n-2)\sqrt{n(n-1)}.$$

2. Performing $P_{DSS}(\Gamma_G)\ (\lambda) = 0$ from Theorem 3.2 (2) for even n, we get the eigenvalues of Γ_G, which are $\lambda_1 = 0$ of multiplicity $2n - 4$, and the other two eigenvalues are $\lambda_{2,3} = \pm\, i(3n - 4)(n - 4)\sqrt{n(n-2)}$. So that

$$Spec(\Gamma_G) = \left\{\left(i(3n-4)(n-4)\sqrt{n(n-2)}\right)^1, (0)^{2n-4}, \left(-i(3n-4)(n-4)\sqrt{n(n-2)}\right)^1\right\}.$$

From the spectrum as mentioned earlier and following Equation (3), we finally arrive at

$$\rho_{DSS}(\Gamma_G) = (3n-4)(n-4)\sqrt{n(n-2)}.$$

Theorem 3.4 In Γ_G for $G = G_1 \cup G_2 \subset D_{2n}$,

(1) $E_{DSS}(\Gamma_G) = 2(3n-2)(n-2)\sqrt{n(n-1)}$, for odd n,

(2) $E_{DSS}(\Gamma_G) = 2(3n-4)(n-4)\sqrt{n(n-2)}$, for even n.

Proof.
1. From the eigenvalues of *Spec*(Γ_G) in Theorem 3.3 (1) for odd n, we can obtain the *DSS*-energy of Γ_G. Since $n \geq 3$ for $n \in \mathbb{N}$ and n is odd, then $3n^2 - 8n + 4$ is always positive. According to the formula on Equation (2), therefore,

$$E_{DSS}(\Gamma_G) = (2n-3)|0| + \left|\pm i(3n^2-8n+4)\sqrt{n(n-1)}\right|$$

$$= \sqrt{(3n^2-8n+4)^2(n(n-1))} + \sqrt{(-(3n^2-8n+4))^2(n(n-1))}$$

$$= 2(3n^2-8n+4)\sqrt{n(n-1)}$$

$$= 2(3n-2)(n-2)\sqrt{n(n-1)}.$$

2. For even n, it follows from Theorem 3.3 (2) and Equation (2), the *DSS*-energy is presented herewith

$$E_{DSS}(\Gamma_G) = (2n-4)|0| + \left|\pm i(3n^2-16n+16)\sqrt{n(n-2)}\right|$$

$$= \sqrt{(3n^2-16n+16)^2(n(n-2))} + \sqrt{(-(3n^2-16n+16))^2(n(n-2))}$$

$$= 2(3n^2-16n+16)^2\sqrt{n(n-2)}$$

$$= 2(3n-4)(n-4)\sqrt{n(n-2)}.$$

Note that $3n^2 - 16n + 16$ is always nonnegative for $n \geq 3$, $n \in \mathbb{N}$.

DISCUSSION

By examining the results of Theorems 3.3 and 3.4, we obtain the explicit fact that the degree square subtraction energy is twice the spectral radius of Γ_G.

Corollary 4.1 In Γ_G for $G = G_1 \cup G_2 \subset D_{2n}$, $E_{DSS}(\Gamma_G) = 2 \cdot \rho_{DSS}(\Gamma_G)$.

Theorem 3.4 allows us to classify the degree square subtraction energy of Γ_G for D_{2n} based on Equation (4). It is presented herewith.

Corollary 4.2 For $G = G_1 \cup G_2 \subset D_{2n}$, Γ_G is hyperenergetic associated with the degree square subtraction matrix.

Furthermore, based on the findings showed in Theorem 3.4, we derive the following fact.

Corollary 4.3 In Γ_G for $G = G_1 \cup G_2 \subset D_{2n}$, the degree square subtraction energy for Γ_G is never an odd integer.

Moreover, Romdhini and Nawawi (2013) have formulated the degree subtraction energy (DSt) energy of Γ_G for D_{2n} as presented in Equation (1), and here we can conclude the relationship.

Corollary 4.4 In Γ_G for $G = G_1 \cup G_2 \subset D_{2n}$, then

1. $E_{DSS}(\Gamma_G) = (3n-2)\, E_{DSt}(\Gamma_G)$, for odd n, and

2. $E_{DSS}(\Gamma_G) = (3n-4)\, E_{DSt}(\Gamma_G)$, for even n.

It is shown that the degree square subtraction energy is always greater than the degree subtraction energy of Γ_G for D_{2n}.

CONCLUSION

We showed the spectral radius of Γ_G associated with the degree square subtraction matrix. We then presented the degree square subtraction energy of Γ_G for D_{2n}. We also presented the degree square subtraction energy which is twice the spectral radius of Γ_G. Moreover, it is also observed that there is a relationship between degree square subtraction energy and degree subtraction energy that was reported in previous literature.

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