Sains Malaysiana 53(11)(2024): 3803-3815
http://doi.org/10.17576/jsm-2024-5311-22

A Linearization based on Taylor Expansion to Multi-Objective Linear Fractional Program with Fuzzy Coefficients and Fuzzy Decision Variables

(Linearisasi berdasarkan Pengembangan Taylor kepada Program Pecahan Linear Pelbagai Objektif dengan Pekali Kabur dan Pemboleh Ubah Keputusan Kabur)

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Received: 23 January 2024/Accepted: 27 September 2024

ABSTRACT

Reaching an efficient solution for multi-objective programming problem (MOPP) is not easy and may encompass some hardships due to existing more than one objective. The aim of this research was to introduce a new efficient method to tackle fully fuzzy multi-objective linear fractional programming problem (FFMOLFPP) i.e., a multi-objective linear fractional programming problem (MOLFPP) with fuzzy coefficients and fuzzy decision variables. To construct the approach, the $\alpha - cuts$ of the fuzzy numbers, variable transformations, the first-order Taylor series, the membership functions, and the weighted sum method are used. In two phases, this method alters the fully fuzzy problem into linear programming problem (LPP) which its solution is at least a weakly ϵ −efficient for the main problem. Numerical examples are compared to an existing method and the outcomes demonstrate that our proposed method is much more accurate.

Keywords: Fuzzy numbers; membership functions; Taylor series; the weighted sum method

ABSTRAK

Mencapai penyelesaian yang cekap untuk masalah pengaturcaraan berbilang objektif (MOPP) bukanlah mudah dan mungkin merangkumi beberapa kesukaran kerana terdapat lebih daripada satu objektif sedia ada. Matlamat penyelidikan ini adalah untuk memperkenalkan kaedah baharu yang cekap untuk menangani masalah pengaturcaraan pecahan linear berbilang objektif kabur sepenuhnya (FFMOLFPP) iaitu masalah pengaturcaraan pecahan linear berbilang objektif (MOLFPP) dengan pekali kabur dan pemboleh ubah keputusan kabur. Untuk membina pendekatan ini, potongan α nombor kabur, transformasi pemboleh ubah, siri Taylor tertib pertama, fungsi keahlian dan kaedah jumlah wajaran digunakan. Dalam dua fasa, kaedah ini mengubah masalah kabur sepenuhnya kepada masalah pengaturcaraan linear (LPP) yang penyelesaiannya sekurang-kurangnya ϵ-cekap untuk masalah utama. Contoh berangka dibandingkan dengan kaedah sedia ada dan hasilnya menunjukkan bahawa kaedah cadangan kami adalah lebih tepat.

Kata kunci: Fungsi keahlian; kaedah hasil tambah wajaran; nombor kabur; siri Taylor

INTRODUCTION

Linear fractional programming problem (LFPP) and multi-objective programming are modeled in optimization widely. Some applications of LFPP in different disciplines such as in economy, business, engineering, and management were demonstrated by Stancu-Minasian (1997). Furthermore, LFPP can be used as an appropriate model in transportation, water consumption, medicine, and industry (Ahmad et al. 2020; Das, Edalatpanah & Mandal 2020; Radhakrishnan & Anukokila 2017; Wang et al. 2019). Besides, Vafamand et al. (2021, 2020) developed the application of multi-objective controllers in medical science and industry; Mahmoodirad, Garg and Niroomand (2022) transformed a set covering problem, which has application in the real world problems such as facility problems, and airlines schedules problem into the MOLFPP; Garg, Mahmoodirad and Niroomand (2021) studied a fractional two-stage transshipment problem where all the parameters are represented by fuzzy numbers; Garai and Garg (2019) presented a multi-objective linear fractional inventory problem with generalized intuitionistic fuzzy numbers.

In different fields of optimization such as engineering, business, and management, the notion of fuzzy sets has been used to design approaches (Borza, Rambely & Saraj 2012; Rashmanlou & Borzooei 2016; Zapata et al. 2020). Specifically, one can use fuzzy numbers when there exists an ambiguity to specify coefficients. In LFPP, we cope with the fuzzy linear fractional programming problem (FLFPP) if the parameters are fuzzy numbers. One way of solving FLFPP is to use fuzzy ranking approaches. In this approach, a fuzzy number is altered into fixed numbers. Consequently, multiple LFPPs are created instead of the main fuzzy problem (Arya et al. 2020). Although these kinds of approaches are easy and straightforward, replacing a fuzzy number with fixed numbers may not be as comprehensive as we expected generally. On the other hand, using the notion of $\alpha - cuts$ has been considered by many researchers as an efficient and comprehensive method tackling the fuzzy numbers (Rao 2017). Overall, when the concept of $\alpha - cuts$ is utilized, maximization of the FLFPP subject to feasible region S can be reduced into:

$$\underset{X \in S}{\text{Maximize}} \{F^L(X), F^U(X)\}, \tag{I}$$

where $F^L(X)$ and $F^U(X)$ are linear fractional objectives functions.

A number of methods have been introduced to address problem (I) (Borza & Rambely 2023, 2022a; Borza, Rambely & Edalatpanah 2023; Chinnadurai & Mathukumar 2016; Mehra, Chandra & Bector 2007; Stanojevic & Stanojevic 2013; Veeramani & Sumathi 2014). Moreover, there are several approaches to deal with MOLFPP which can be also utilized to tackle problem (I) (Borza & Rambely 2021a; Chakraborty & Gupta 2002; De & Deb 2015; Pal, Moitra & Maulik 2003; Toksari 2008). In addition, the weighted sum approach can be utilized and then change the problem (I) into the sum of linear fractional programming problem (S-LFPP). In this case, recent works of Borza and Rambely (2021b), a non-iterative algorithm based on variable transformations, and Liu et al. (2019), an iterative method based on a branch and bound algorithm, are useful.

In MOLFPP, if the coefficients are fuzzy numbers, then we face fuzzy multi-objective linear fractional programming problem (FMOLFPP). The most recent algorithms dealing with this problem were proposed by Borza and Rambely (2022b) and Nayak and Ojha (2019). In Nayak and Ojha, the fuzzy problem is finally changed into LPP using the $\alpha - cuts$ of the fuzzy numbers, and the first Taylor expansion. To be more precise, in their approach, the fuzzy problem is firstly changed into:

$$\underset{X \in S}{\text{Maximize}} \left\{ [F_i^L(X), F_i^U(X)] = \left[\frac{C_i^T X + d_i}{P_i^T X + q_i}, \frac{E_i^T X + f_i}{G_i^T X + h_i} \right], i = 1, \dots, k \right\}. \tag{II}$$

Let X_i^* be the optimal solution of $\underset{X \in S}{\text{Maximize}} F_i^U(X)$; and $\breve{F}_i^L(X)$ be the first-order Taylor expansion of $F_i^U(X)$ around $X_i^*, i = 1, \dots, k$. In their method, the following problem is eventually solved.

$$\underset{X \in S}{\text{Maximize}} \sum_{i=1}^{k} w_i \breve{F}_i^U(X) \tag{III}$$

There is a weakness in the method that is the only use of $F_i^U(X)$. In other word, $F_i^L(X), i = 1, \dots, k$ is not considered finding the solution. In Borza and Rambely (2022b), they proposed an algorithm based on a parametric approach of Dinkelbach (1967), method of Mehra, and the concept of $\alpha - cuts$. In their method, $F_i^U(X)$ plays a more important role than $F_i^L(X)$ which can be considered as a drawback.

The concept of fully fuzzy programming problem arises if both coefficients and decision variables are fuzzy numbers. Researchers has proposed different algorithms coping with fully fuzzy LPP and fully fuzzy LFPP (Borza & Rambely 2022c; Deb 2018; Kumar, Kaur & Singh 2011). To the best of our knowledge, FFMOLFPP has been only considered by Arya et al. In the method, the problem is transformed into a problem with deterministic parameters using a fuzzy ranking approach. Of course, using the fuzzy ranking approach ease coping with the problem since a fuzzy parameter, which includes countless numbers, is finally represented by three fixed numbers. However, this technique may not cover all the possibilities and therefore cannot be as much comprehensive as the decision maker expects. The weakness of their method is demonstrated in numerical example section. Apart from that, their method is only applicable for the problem with triangular fuzzy numbers.

This paper aims to design a new method to address FFMOLFPP which overcomes the drawbacks of Arya et al. We construct our approach in two phases, where a new technique based on variable transformations is introduced to deal with fully fuzzy linear fractional programming problem (FFLFPP) in phase 1. In the second phase, FFMOLFPP is considered and transformed into interval valued MOLFPP by the use of $\alpha - cuts$. Subsequently, taking into account the first phase, individual problems are solved and then the individual interval-valued linear fractional objective functions are linearized using the first-order Taylor expansions about the individual optimums. Therefore, the main fuzzy problem is changed into interval-valued multi-objective linear programming problem (I-VMOLPP) which is altered into LPP applying the weighted sum technique twice. It should be mentioned, in order to normalize the linear objectives, the concept of the membership functions was utilized in the approach. Examples are taken from Arya et al. to illustrate the method and comparisons indicate the superiority of this study's outcomes. In addition, the approach is not limited to any specific kind of fuzzy numbers.

The paper is organized as follows. Some preliminaries are given in the next section. The main outcomes are released subsequently. Indeed, it is demonstrated that FFMOLFPP is altered into LPP. After that, numerical examples are given to illustrate the method, and comparisons are made to show the accuracy. Finally, last section concludes the paper.

PRELIMINARIES

FUZZY NUMBERS AND INTERVALS

Definition 1 (Wang 1996) Let $\tilde{A}$ be a normalized fuzzy set. A triangular fuzzy number $\tilde{A}$ is defined as:

$$\mu_{\tilde{A}}(x) = \mu_{\tilde{A}}(x;a,b,c) = \begin{cases} (x-a)/(b-a), & x \in [a,b) \\ (c-x)/(c-b), & x \in [b,c] \\ 0, & x > c \text{ or } x < a. \end{cases}$$

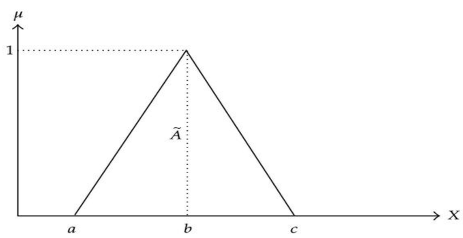

FIGURE 1. Triangular fuzzy number $\tilde{A}$

Definition 2 (Wang 1996) Let $\tilde{A}$ be a fuzzy set in X and $\alpha \in [0,1]$. The $\alpha - cuts$ of the fuzzy set $\tilde{A}$ is the crisp set $\tilde{A}_\alpha$ given by: $[\tilde{A}]_\alpha = \{x \in X: \mu_{\tilde{A}}(x) \geq \alpha\}$.
Let $\tilde{A}$ be a triangular fuzzy number with the membership function $\mu_{\tilde{A}}(x;a,b,c)$, then $[\tilde{A}]_\alpha = [a+\alpha(b-a),\ c-\alpha(c-b)]$.
Furthermore, let $\tilde{V}^T = (\tilde{v}_1, \dots, \tilde{v}_n)$ be a fuzzy vector i.e., a vector with fuzzy elements, then, $[\tilde{V}]_\alpha^T = ([\tilde{v}_1]_\alpha, \dots, [\tilde{v}_n]_\alpha)$.

Definition 3(Wang 1996) Let $\tilde{Z}$ be a fuzzy set in X and $\alpha \in [0,1]$ with $[\tilde{Z}]_\alpha = [Z_\alpha^-, Z_\alpha^+]$. Then, the *center average defuzzifier* of $\tilde{Z}$ is given by: Dfuz$([\tilde{Z}]_\alpha) = \frac{Z_\alpha^- + Z_\alpha^+}{2}$.

Definition A (Ranking of fuzzy numbers) Let $\tilde{A}, \tilde{B}, \tilde{C}$ be fuzzy numbers with $\alpha - cuts$ $[\tilde{A}]_\alpha = [a_\alpha^-, a_\alpha^+]$, $[\tilde{B}]_\alpha = [b_\alpha^-, b_\alpha^+]$, and $[\tilde{C}]_\alpha = [c_\alpha^-, c_\alpha^+]$. According to Kaufmann and Gupta (1988), possibility and necessity theories can be used to rank fuzzy numbers based on their $\alpha - cuts$ as follows:

Method 1 We say $\tilde{A}$ is smaller than $\tilde{B}$, and denoted by $\tilde{A} \leq \tilde{B}$, if and only if $a_\alpha^- \leq b_\alpha^-$, and $a_\alpha^+ \leq b_\alpha^+$ for $\alpha \in (0,1]$.
Moreover, from Zimmermann (2001), for $k_1, k_2 \geq 0$, we say $k_1\tilde{A} + k_2\tilde{B} \leq \tilde{C}$, if and only if $k_1 a_\alpha^- + k_2 b_\alpha^- \leq c_\alpha^-$, and $k_1 a_\alpha^+ + k_2 b_\alpha^+ \leq c_\alpha^+$.

Method 2 We say $\tilde{A}$ is smaller than $\tilde{B}$, and denoted by $\tilde{A} \leq \tilde{B}$, if and only if $a_\alpha^+ \leq b_\alpha^+$ for $\alpha \in (0.5, 1]$.
Furthermore, for $k_1, k_2 \geq 0$, we say $k_1\tilde{A} + k_2\tilde{B} \leq \tilde{C}$, if and only if $k_1 a_\alpha^+ + k_2 b_\alpha^+ \leq c_\alpha^+$.
Notify that in spite of method 1, method 2 can be applied to rank any two fuzzy numbers. However, method 2 is weaker since only the upper bounds of the intervals are utilized. Therefore, in this paper, we use method 1 as long as this method works successfully. Otherwise, method 2 is examined.

Definition 5 (Moore, Kearfott & Cloud 2009) Assume that $A = [A^L, A^U]$, $B = [B^L, B^U]$, and $k \geq 0$ is a scalar. Therefore, addition, multiplication, and division on the intervals are defined as follows:

$$A+B = [A^L + B^L, A^U + B^U],\ -A = [-A^U, -A^L],\ kA = [kA^L, kA^U],$$
$$AB = [\min\{A^LB^L, A^LB^U, A^UB^L, A^UB^U\},$$
$$\max\{A^LB^L, A^LB^U, A^UB^L, A^UB^U\}],$$
$$\frac{A}{B} = \left[\min\left\{\frac{A^L}{B^L}, \frac{A^L}{B^U}, \frac{A^U}{B^L}, \frac{A^U}{B^U}\right\}, \max\left\{\frac{A^L}{B^L}, \frac{A^L}{B^U}, \frac{A^U}{B^L}, \frac{A^U}{B^U}\right\}\right].$$

LINEAR FRACTIONAL PROGRAMMING

Consider the general form of LFPP as follows:

$$\text{Maximize } \frac{C^TX+p}{D^TX+q} \quad \text{subject to } F = \{AX \leq b, X \geq 0\}, \tag{1}$$

where $A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^{m\times 1}, C, D \in \mathbb{R}^{n\times 1}$, $p, q \in \mathbb{R}$. In addition, $D^TX + q > 0, \forall X = (X_1, \dots, X_n) \in F$. It is additionally assumed that feasible region F is a regular set i.e., a non-empty and bounded set. Notify $X \geq 0$ means $X_i \geq 0$, $i = 1, \dots, n$. Using variable transformations $t = \frac{1}{D^TX+\beta}$, $Y = tX$ changes problem (1) into:

$$\text{Maximize } C^TY + pt \text{ subject to } \phi = \{AY - bt \leq 0,\ D^TY + qt = 1, Y, t \geq 0\}. \tag{2}$$

Lemma 1 (Charnes and Cooper 1962) In problem (2), variable t cannot be zero.

Lemma 2 (Charnes and Cooper 1962) If $(\bar{Y}, \bar{t}) \in \phi$, then $\frac{\bar{Y}}{\bar{t}} \in F$.

Theorem 1 (Charnes and Cooper 1962) If (Y^*, t^*) is optimum for problem (2), then $X^* = \frac{Y^*}{t^*}$ is optimum for problem (1).

MULTI OBJECTIVE PROGRAMMING PROBLEM

The general form of MOPP is as follows:

$$\text{Maximize } \{F_1(X), \dots, F_k(X)\} \quad s.t\ X \in S. \tag{3}$$

Definition 6 (Antunes, Alves & Clímaco 2016). In problem (3), a solution $X^* \in S$ is called *efficient* if and only if $\nexists X \in S$ such that $F_j(X^*) \le F_j(X)$, $j = 1, \ldots, k$, and $\exists l \in \{1, \ldots, k\}$ such that $F_l(X^*) < F_l(X)$.

Definition 7 In problem (3), $X^* \in S$ is an $\epsilon -$ efficient solution if $\forall X \in S$, $\exists j \in \{1, \ldots, k\}$ such that $F_j(X^*) - \epsilon < F_j(X)$, where $\epsilon = \inf\{\delta > 0 : F_j(X^*) - \delta < F_j(X)\}$.

The weighted sum approach can be used as a classical method to address the problem (3) as follows:

$$\underset{X \in S}{\text{Maximize}} \sum_{i=1}^{k} w_i F_i(X), \tag{4}$$

where $w_i \ge 0, i = 1, \ldots, k$ and $\sum_{i=1}^{k} w_i = 1$.

Remark 1 Consider the general form of an interval-valued fractional programming problem as follows:

$$\underset{X \in S}{\text{Maximize}} \ \frac{[T(X),\ P(X)]}{[H(X),\ K(X)]} \tag{5}$$

where $H(X), K(X) > 0$, $\forall X \in$ any feasible region S.

Lemma 1 If $\left(\underset{X \in S}{\text{Minimize}}\, T(X)\right) \ge 0$, then problem (5) is changed into:

$\underset{X \in S}{\text{Maximize}} \left[\frac{T(X)}{K(X)}, \frac{P(X)}{H(X)}\right]$.

Proof Obviously, there must exist unique functions $L(X)$ and $U(X)$ such that $\frac{[T(X),\ P(X)]}{[H(X),\ K(X)]} = [L(X), U(X)]$, where $L(X) \le U(X), \forall X \in S$.

Since $\left(\underset{X \in S}{\text{Minimize}}\, T(X)\right) \ge 0$, then $T(X) \ge 0, \forall X \in S$. Thus, $\frac{T(X)}{K(X)} \le \frac{P(X)}{H(X)}, \forall X \in S$ due to the fact that $0 \le T(X) \le P(X)$ and $0 < H(X) \le K(X)$. Therefore, $L(X) = \frac{T(X)}{K(X)}$ and $U(X) = \frac{P(X)}{H(X)}$. The proof is then complete. ▪

Lemma 2 If $\left(\underset{X \in S}{\text{Maximize}}\, P(X)\right) \le 0$, then problem (5) is transformed into:

$\underset{X \in S}{\text{Maximize}} \left[\frac{T(X)}{H(X)}, \frac{P(X)}{K(X)}\right]$.

Lemma 3 If $\left(\underset{X \in S}{\text{Minimize}}\, T(X)\right) \le 0$ and $\left(\underset{X \in S}{\text{Maximize}}\, P(X)\right) \ge 0$, then problem (5) is altered into:

$\underset{X \in S}{\text{Maximize}} \left[\frac{T(X)}{H(X)}, \frac{P(X)}{H(X)}\right]$.

In a similar manner, these two lemmas can be proved. Therefore, the proofs are omitted.

Lemma 4 Let $H(X) = [F(X), G(X)]$ be an interval valued function defined on domain D, where $F(X) \le G(X), \forall X \in D$. Then, $\frac{\partial H(X)}{\partial X}|_{X_0} = \left[min\left\{\frac{\partial F(X)}{\partial X}|_{X_0}, \frac{\partial G(X)}{\partial X}|_{X_0}\right\}, max\left\{\frac{\partial F(X)}{\partial X}|_{X_0}, \frac{\partial G(X)}{\partial X}|_{X_0}\right\}\right]$. *Proof* $H(X) = [F(X), G(X)] \Longrightarrow H(X) = \lambda F(X) + (1-\lambda) G(X), \forall \lambda \in [0,1]$. Thus, $\frac{\partial H(X)}{\partial X}|_{X_0} = \lambda \frac{\partial F(X)}{\partial X}|_{X_0} + (1-\lambda)\frac{\partial G(X)}{\partial X}|_{X_0} = \left[\frac{\partial F(X)}{\partial X}|_{X_0}, \frac{\partial G(X)}{\partial X}|_{X_0}\right]$ if $\frac{\partial F(X)}{\partial X}|_{X_0} \le \frac{\partial G(X)}{\partial X}|_{X_0}$. Otherwise, $\frac{\partial H(X)}{\partial X}|_{X_0} = \left[\frac{\partial G(X)}{\partial X}|_{X_0}, \frac{\partial F(X)}{\partial X}|_{X_0}\right]$. ▪

MAIN RESULTS

PHASE 1: A METHOD TO FFLFPP

In this section, FFLFPP is considered and eventually transformed into LPP. To reach this aim, the concept of $\alpha - cuts$, variable transformations, the weighted sum approach, and the first-order Taylor expansion are used. Consider the general form of FFLFPP as follows:

$$\begin{aligned} &\text{Maximize } \tilde{F}(\tilde{X}) = \frac{\tilde{C}^T \tilde{X} + \tilde{d}}{\tilde{M}^T \tilde{X} + \tilde{n}}, \\ &\text{s.t } \tilde{S} = \{\tilde{A}\tilde{X} \le \tilde{b}, \tilde{X} \ge 0\}, \end{aligned} \tag{6}$$

where $\tilde{M}^T \tilde{X} + \tilde{n} > 0, \forall \tilde{X} = (\tilde{X}_1, \ldots, \tilde{X}_n) \in \tilde{S}$.

The use of $\alpha - cuts$ transforms problem (6) into:

$$\begin{aligned} &\text{Maximize } \bar{F}(X^L, X^U) = \frac{[C^L,\ C^U][X^L,\ X^U] + [d^L,\ d^U]}{[M^L,\ M^U][X^L,\ X^U] + [n^L,\ n^U]} \\ &\text{s.t } \bar{S} = \{[A^L, A^U][X^L, X^U] \le [b^L, b^U], \\ &\quad X^L \le X^U,\ X^L, X^U \ge 0\}, \end{aligned} \tag{7}$$

where $[X^L, X^U] = ([X_1^L, X_1^U], \ldots, [X_n^L, X_n^U]) = (X_1^L, X_1^U, \ldots, X_n^L, X_n^U) = (X^L, X^U)$, and $[M^L, M^U][X^L, X^U] + [n^L, n^U] > 0, \forall (X^L, X^U) \in \bar{S}$.

For convenience, without loss of generality, let us assume that:

$C^L X^L + d^L = min\{C^L X^L + d^L, C^L X^L + d^U, C^L X^U + d^L, C^L X^U + d^U, C^U X^L + d^L, C^L X^U + d^U, C^U X^L + d^L, C^U X^L + d^U, C^U X^U + d^L, C^U X^U + d^U\}$,

$C^U X^U + d^U = max\{C^L X^L + d^L, C^L X^L + d^U, C^U X^U + d^L, C^L X^U + d^U, C^U X^L + d^L, C^U X^L + d^U, C^U X^U + d^L, C^U X^U + d^U\}$,

$M^L X^L + n^L = min\{M^L X^L + n^L, M^L X^L + n^U, M^L X^U + n^L, M^L X^U + n^U, M^U X^L + n^L, M^U X^L + n^U, M^U X^U + n^L, M^U X^U + n^U\}$,

$M^U X^U + n^U = max\{M^L X^L + n^L, M^L X^L + n^U, M^L X^U + n^L, M^L X^U + n^U, M^U X^L + n^L, M^U X^L + n^U, M^U X^U + n^L, M^U X^U + n^U\}$,

$min\{A^L X^L, A^L X^U, A^U X^L, A^U X^U\} = A^L X^L$, and

$max\{A^L X^L, A^L X^U, A^U X^L, A^U X^U\} = A^U X^U$, $\quad \forall (X^L, X^U) \in \bar{S}$.

On these assumptions and also applying the ranking method 1 and the arithmetic of intervals, problem (7) is changed into:

$$\text{Maximize } \bar{F}(X^L,X^U)=\frac{[C^LX^L+d^L,\ C^UX^U+d^U]}{[M^LX^L+n^L,\ M^UX^U+n^U]}$$
$$\text{s.t } S=\{A^LX^L\le b^L, A^UX^U\le b^U, X^L\le X^U, \quad (8)$$
$X^L,X^U\ge 0\}$, where $M^UX^U+n^U>0, \forall (X^L,X^U)\in S$.

Setting $t=\frac{1}{[M^LX^L+n^L,\ M^UX^U+n^U]}$, $Y^L=tX^L, Y^U=tX^U$ alters problem (8) into:

$$\text{Maximize}$$
$$\bar{F}(Y^L,Y^U)=[C^LY^L+td^L,\ C^UY^U+td^U] \quad (9)$$
$$\text{s.t}, [M^LY^L+tn^L,\ M^UY^U+tn^U]=1\}.$$

In problem (9), we can replace the constraints $[M^LY^L+tn^L,\ M^UY^U+tn^U]=1=[1,1]$ with the two constraints $[M^LY^L+tn^L,\ M^UY^U+tn^U]=1=[1,1]$and $[1,1]\le[M^LY^L+tn^L,\ M^UY^U+tn^U]$, simultaneously. If we apply the ranking method 1, then these two constraints results in:
$M^LY^L+tn^L=M^UY^U+tn^U=1$; this means the original problem must be a LPP, which is a contradiction. Moreover, if we apply ranking method 2, then $[M^LY^L+tn^L,\ M^UY^U+tn^U]=1$ is changed into: $M^UY^U+tn^U=1$. We know that the existence of an equality constraint either makes the problem infeasible or limits the feasible region so much that the solution obtained may not be optimum for the original problem. To overcome these difficulties and tackle the interval constraint, the following lemma is proved.

Lemma 5 $[a,b]=1$ if and only if $a\le 1, b\ge 1$.
Proof If $[a,b]=1$, then $a=\lambda a+(1-\lambda)a\le\lambda a+(1-\lambda)b=[a,b]=1$,
$1=[a,b]=\lambda a+(1-\lambda)b\le\lambda b+(1-\lambda)b=b$.If $a\le 1, b\ge 1$, then the ranking method 1 results:$[a,b]\nless[1,1]$ and $[a,b]\ngtr[1,1]$; this means $[a,b]=[1,1]=1$. ■

Taking into account Lemma 5 changes problem (9) into:

$$\text{Maximize} \quad C^UY^U+td^U]$$
$$\text{s.t } \Omega=\{A^LY^L-tb^L\le 0, A^UY^U-tb^U\le 0,$$
$$Y^L\le Y^U,\ Y^L,Y^U,t\ge 0, \quad (10)$$
$$M^LY^L+tn^L\le 1,\ M^UY^U+tn^U\ge 1\}.$$

According to Chanas and Kutcha (1994), problem (10) is changed into:

$$\text{Maximize } \{C^LY^L+td^L,\ C^UY^U+td^U\}$$
$$\text{s.t } (Y^L,Y^U,t)\in\Omega. \quad (11)$$

Problem (11) is a bi-objective LPP and can be solved using several methods such as the weighted sum method and the max-min technique.
The weighted sum approach with equal weights transforms problem (11) into:

$$\underset{(Y^L,Y^U,t)\in\Omega}{\text{Maximize}} \quad C^LY^L+C^UY^U+(d^L+d^U)t \quad (12)$$

Let problem (12) is solved and the optimal solution is $(Y^{L*},Y^{U*},t^*,\beta^*)$. Therefore, the solution proposed for problem (8) and consequently for problem (6) is: $[\tilde{X}^*]_\alpha=[X^{L*},X^{U*}]=\left[\frac{Y^{L*}}{t^*},\frac{Y^{U*}}{t^*}\right]$.

PHASE 2: A METHOD TO FFMOLFPP

Consider the general form of FFMOLFPP as follows:

$$\text{Maximize}\left\{\tilde{F}_i(\tilde{X})=\frac{\tilde{C}_i^T\tilde{X}+\tilde{d}_i}{\tilde{M}_i^T\tilde{X}+\tilde{n}_i}, i=1,\dots,k\right\}$$
$$\text{s.t } \tilde{S}=\{\tilde{A}\tilde{X}\le\tilde{b},\tilde{X}\ge 0\}. \quad (13)$$

It is obvious that the method introduced in Phase 1 can be applied to the following individual problem.

$$\underset{\tilde{X}\in\tilde{S}}{\text{Maximize}}\ \tilde{F}_i(\tilde{X}), i=1,\dots,k. \quad (14)$$

In Phase 1, it was demonstrated that using the concept of the $\alpha-cuts$ transforms problem (14) into:

$$\bar{F}_i(X^L,X^U)=\frac{\left[\underline{f_i}(X^L,X^U),\ \bar{f}_i(X^L,X^U)\right]}{\left[\underline{g_i}(X^L,X^U),\ \bar{g}_i(X^L,X^U)\right]} \quad (15)$$
$$\text{s.t } S=\{A^LX^L\le b^L, A^UX^U\le b^U,$$

where $\underline{f_i}$, $\bar{f}_i$, $\underline{g_i}$, $\bar{g}_i$ are linear functions and $\underline{f_i}(X^L,X^U)\le\bar{f}_i(X^L,X^U)$, $\underline{g_i}(X^L,X^U),\le\bar{g}_i(X^L,X^U), \forall(X^L,X^U)\in S$, $i=1,\dots,k$.

Let the method proposed in Phase 1 solves problem (15) and the solution obtained is:
$[\tilde{X}^*]^i_\alpha=[X^{*iL},X^{*iU}]$, for $i=1,\dots,k$.

Now, using the first-order Taylor series about $[\tilde{X}^*]^i_\alpha$ linearizes the objective function $\bar{F}_i(X^L,X^U)$ as follows:

$$\text{Linearization}\left(\bar{F}_i(X^L,X^U)\right)=$$
$$\frac{\partial\bar{F}_i(X^L,X^U)}{\partial X^L}\Big|_{[\tilde{X}^*]^i_\alpha}(X^L-X^{*iL})+\frac{\partial\bar{F}_i(X^L,X^U)}{\partial X^U}\Big|_{[\tilde{X}^*]^i_\alpha}(X^U-X^{*iU})+\bar{F}_i(X^{*iL},X^{*iU})= \quad (16)$$
$$[\underline{\text{Lin}}\,\bar{F}_i(X^L,X^U),\overline{\text{Lin}}\,\bar{F}_i(X^L,X^U)].$$

In order to reach a satisfactory solution, the objectives are normalized in MOPP. In this study, we use the concept of the membership function to reach this aim. To do this, let us assume:

$\underline{F}_i^{min} = \underset{X\in S}{\text{Minimize}}\ \underline{\text{Lin}}\,\overline{F}_i(X^L, X^U)$,

$\underline{F}_i^{max} = \underset{X\in S}{\text{Maximize}}\ \underline{\text{Lin}}\,\overline{F}_i(X^L, X^U)$,

$\underline{F}_i^{max} = \underset{X\in S}{\text{Maximize}}\ \underline{\text{Lin}}\,\overline{F}_i(X^L, X^U)$, and

$\overline{F}_i^{min} = \underset{X\in S}{\text{Minimize}}\ \overline{\text{Lin}}\,\overline{F}_i(X^L, X^U)$,

$\overline{F}_i^{max} = \underset{X\in S}{\text{Maximize}}\ \overline{\text{Lin}}\,\overline{F}_i(X^L, X^U)$.

Therefore, the membership functions can be specified as follows:

$$\mu_{\underline{\text{Lin}}\,\overline{F}_i}(X^L, X^U) = \frac{1}{\underline{F}_i^{max} - \underline{F}_i^{min}}\left(\underline{\text{Lin}}\,\overline{F}_i(X^L, X^U) - \underline{F}_i^{min}\right),$$
and
$$\mu_{\underline{\text{Lin}}\,\overline{F}_i}(X^L, X^U) = \frac{1}{\underline{F}_i^{max} - \underline{F}_i^{min}}\left(\underline{\text{Lin}}\,\overline{F}_i(X^L, X^U) - \underline{F}_i^{min}\right), \quad (17)$$
$i = 1, \dots, k.$

Accordingly, problem (13) is substituted by:

$$\text{Maximize}\ \left\{\left[\mu_{\underline{\text{Lin}}\,\overline{F}_i}(X^L, X^U),\ \mu_{\overline{\text{Lin}}\,\overline{F}_i}(X^L, X^U)\right],\ i = 1, \dots, k\right\}$$
$$\text{s.t } S = \{A^L X^L \le b^L, A^U X^U \le b^U,\ X^L \le X^U,\ X^L, X^U \ge 0\}. \quad (18)$$

Using the weighted sum technique transforms problem (18) into:

$$\underset{(X^L,X^U)\in S}{\text{Maximize}} \sum_{i=1}^{k} w_i \left[\mu_{\underline{\text{Lin}}\,\overline{F}_i}(X^L, X^U), \mu_{\overline{\text{Lin}}\,\overline{F}_i}(X^L, X^U)\right] = \left[\sum_{i=1}^{k} w_i \times \mu_{\underline{\text{Lin}}\,\overline{F}_i}(X^L, X^U), \sum_{i=1}^{k} w_i \times \mu_{\overline{\text{Lin}}\,\overline{F}_i}(X^L, X^U)\right], \quad (19)$$

where $w_i \ge 0, i = 1, \dots, k$ are determined by the decision maker based on the importance of the objective functions. According to Chanas and Kutcha, problem (19) is changed into:

$$\underset{(X^L,X^U)\in S}{\text{Maximize}} \left\{\sum_{i=1}^{k} w_i \times \mu_{\underline{\text{Lin}}\,\overline{F}_i}(X^L, X^U), \sum_{i=1}^{k} w_i \times \mu_{\overline{\text{Lin}}\,\overline{F}_i}(X^L, X^U)\right\}. \quad (20)$$

Problem (20) is a bi-objective linear programming problem and the weighted sum approach with equal weights transforms it into a LPP as follows:

$$\text{Maximize}\ \sum_{i=1}^{k} w_i \times \mu_{\underline{\text{Lin}}\,\overline{F}_i}(X^L, X^U) + \sum_{i=1}^{k} w_i \times \mu_{\overline{\text{Lin}}\,\overline{F}_i}(X^L, X^U) \quad (21)$$
$$\text{s.t } (X^L, X^U) \in S.$$

ACCURACY CALCULATION AND DEFUZZIFICATION

According to Remark 1, we can set:

$$\overline{F}_i(X^L, X^U) = [F_{li}(X^L, X^U), F_{ui}(X^L, X^U)] = \left[\frac{f_i(X^L, X^U)}{h_i(X^L, X^U)}, \frac{g_i(X^L, X^U)}{k_i(X^L, X^U)}\right].$$

Let us assume that $\underline{X}^{*i} = (\underline{X}^{*iL}, \underline{X}^{*iU})$ and are the optimal solutions to $\underset{(X^L,X^U)\in S}{\text{Maximize}} \frac{f_i(X^L, X^U)}{h_i(X^L, X^U)}$ and $\underset{(X^L,X^U)\in S}{\text{Maximize}} \frac{g_i(X^L, X^U)}{k_i(X^L, X^U)}$, respectively, $i = 1, \dots, k$.

Now, Let $[\tilde{X}^*]_\alpha = X^* = (X^{*L}, X^{*U})$ be a proposed solution for problem (13), then a criteria to evaluate this solution can be defined as follows:

$$Er(X^*) = \frac{\dfrac{\sum_{i=1}^{k}\left(F_{li}(\underline{X}^{*i}) - F_{li}(X^*)\right)}{|F_{li}(\underline{X}^{*i})|} + \dfrac{\sum_{i=1}^{k}\left(F_{ui}(\overline{X}^{*i}) - F_{ui}(X^*)\right)}{|F_{ui}(\overline{X}^{*i})|}}{2k}. \quad (22)$$

Moreover, an acceptable interval for the value of the i^{th} objective function $[\tilde{F}_i(\tilde{X})]_\alpha$ is: $[F_{li}(X^*), F_{ui}(X^*)]$.

In addition, if we set

$\epsilon_i = max\left\{F_{li}(\underline{X}^{*i}) - F_{li}(X^*), F_{ui}\left(\overline{X}^{*i}\right) - F_{ui}(X^*)\right\}, i = 1, \dots, k$ and $\epsilon = \min\{\epsilon_i, i = 1, \dots, k\}$, then, the solution proposed X^* is at least a weakly ϵ −efficient solution for the main fuzzy problem (13)

Definition 8 For $x, y \in S$, one has xPy (we say y is preferred to x) if $Er(y) \le Er(x)$. Apart from that if we also have $\epsilon(y) \le \epsilon(x)$, then we say y is strongly preferred to x which is denoted by $xPPy$, in this paper.

NUMERICAL EXAMPLE

In this section, three examples taken from Arya et al. are considered and the outcomes are compared to evaluate our approach.

EXAMPLE 1

$$\text{Maximize}\ \{\tilde{F}_1(\tilde{X}), \tilde{F}_2(\tilde{X})\} = \left\{\frac{-(2,\ 3,\ 4)\tilde{X}_1 + (1,\ 2,\ 3)\tilde{X}_2}{(0.5,\ 1,\ 1.5)\tilde{X}_1 + (0.5,\ 1,\ 1.5)\tilde{X}_2 + (2,\ 3,\ 4)}, \frac{(6,\ 7,\ 8)\tilde{X}_1 + (0.5,\ 1,\ 1.5)\tilde{X}_2}{(4,\ 5,\ 6)\tilde{X}_1 + (1,\ 2,\ 3)\tilde{X}_2 + (0.5,\ 1,\ 1.5)}\right\} \quad (23)$$
$$\text{s.t } -(0.5, 1, 1.5)\tilde{X}_1 + (0.5, 1, 1.5)\tilde{X}_2 \le -(0.5, 1, 1.5),$$
$$(1, 2, 3)\tilde{X}_1 + (2, 3, 4)\tilde{X}_2 \le (14, 15, 16),$$
$$-(0.5, 1, 1.5)\tilde{X}_1 \le -(2, 3, 4), \qquad \tilde{X}_1, \tilde{X}_2 \ge 0.$$

Using the concept of $\alpha - cuts$ transforms problem (23) into:

$$\text{Maximize } \{\bar{F}_1(X^L,X^U),\ \bar{F}_2(X^L,X^U)\} = \{\bar{F}_1(X^L,X^U) = \frac{-[2+\alpha,4-\alpha][X_1^L,X_1^U]+[1+\alpha,3-\alpha][X_2^L,X_2^U]}{[0.5+0.5\alpha,1.5-0.5\alpha][X_1^L,X_1^U]+[0.5+0.5\alpha,1.5-0.5\alpha][X_2^L,X_2^U]+[2+\alpha,4-\alpha]}$$
$$\bar{F}_2(X^L,X^U) = \frac{[6+\alpha,8-\alpha][X_1^L,X_1^U]+[0.5+0.5\alpha,1.5-0.5\alpha][X_2^L,X_2^U]}{[4+\alpha,6-\alpha][X_1^L,X_1^U]+[1+\alpha,3-\alpha][X_2^L,X_2^U]+[0.5+0.5\alpha,1.5-0.5\alpha]}\}$$
$$\text{s.t } -[0.5+0.5\alpha,1.5-0.5\alpha][X_1^L,X_1^U]+[0.5+0.5\alpha,1.5-0.5\alpha][X_2^L,X_2^U] \le -[0.5+0.5\alpha,1.5-0.5\alpha],$$
$$[1+\alpha,3-\alpha][X_1^L,X_1^U]+[2+\alpha,4-\alpha][X_2^L,X_2^U] \le [14+\alpha,16-\alpha],$$
$$-[0.5+0.5\alpha,1.5-0.5\alpha][X_1^L,X_1^U] \le -[2+\alpha,4-\alpha],$$
$$X_1^L \le X_1^U, X_2^L \le X_2^U,\ X_1^L,X_1^U,X_2^L,X_2^U \ge 0. \tag{24}$$

Setting $\alpha = 0.5$. and considering the interval arithmetic and ranking method 1 finally transform problem (24) into:

$$\text{Maximize } \{\bar{F}_1(X^L,X^U),\ \bar{F}_2(X^L,X^U)\} =$$
$$\{\bar{F}_1(X^L,X^U) = \frac{[1.5X_2^L-3.5X_1^U, -2.5X_1^L+2.5X_2^U]}{[0.75X_1^L+0.75X_2^L+2.5,\ 1.25X_1^U+1.25X_2^U+3.5]}$$
$$\bar{F}_2(X^L,X^U) = \frac{[6.5X_1^L+0.75X_2^L,\ 7.5X_1^U+1.25X_2^U]}{[4.5X_1^L+1.5X_2^L+0.75,\ 5.5X_1^U+2.5X_2^U+1.25]}\}$$
$$\text{s.t } S_{\alpha=0.5} = \{0.75X_2^L - 1.25X_1^U \le -1.25,\ -0.75X_1^L + 1.25X_2^U \le -0.75, 1.5X_1^L + 2.5X_2^L \le 14.5,$$
$$2.5X_1^U + 3.5X_2^U \le 15.5, -0.75X_1^L \le -2.5,\ -1.25X_1^U \le -3.5,$$
$$X_1^L \le X_1^U,\quad X_2^L \le X_2^U, X_1^L,X_1^U,X_2^L,X_2^U \ge 0\}. \tag{25}$$

In what follows to linearize $\bar{F}_i(X^L,X^U)$, $\underset{(X^L,X^U)\in S_{\alpha=0.5}}{\text{Maximize}}\ \bar{F}_i(X^L,X^U)$ is solved so as to find $[\tilde{X}^*]_\alpha^i = [X^{*iL},X^{*iU}], i = 1,2$.

For $i = 1$, problem (12) is formulated as follows:

$$\text{Maximize } -2.5Y_1^L + 1.5Y_2^L - 3.5Y_1^U + 2.5Y_2^U$$
$$\text{s.t } \{0.75Y_2^L - 1.25Y_1^U + 1.25t \le 0, -0.75Y_1^L + 1.25Y_2^U + 0.75t \le 0,$$
$$1.5Y_1^L + 2.5Y_2^L - 14.5t \le 0, 2.5Y_1^U + 3.5Y_2^U - 15.5t \le 0,$$
$$Y_1^L \le Y_1^U, Y_2^L \le Y_2^U,\ Y_1^L,Y_1^U,Y_2^L,Y_2^U,t \ge 0\} = F$$
$$\cup \{0.75Y_1^L + Y_2^L + 2.5t \le 1, Y_1^U + Y_2^U + 3.5t \ge 1\}. \tag{26}$$

Problem (26) is solved and the solution obtained is:

$(Y_1^{*1L}, Y_1^{*1U}, Y_2^{*1L},\ Y_2^{*1U}, t^{*1})$. $= (0.354, 0.354, 0.1487, 0.1487, 0.1062)$ Therefore,

$[\tilde{X}^*]_\alpha^1 = (X_1^{*1L}, X_1^{*1U}, X_2^{*1L},, X_2^{*1U}) = \frac{1}{t^{*1}}(Y_1^{*1L}, Y_1^{*1U}, Y_2^{*1L},\ Y_2^{*1U}) =$ $(3.3333, 3.3333, 1.4, 1.4)$.

For $i = 2$, problem (12) is formed as follows:

$$\text{Maximize } 6.5Y_1^L + 0.75Y_2^L + 7.5Y_1^U + 1.25Y_2^U$$
$$\text{s.t } F \cup \{4.5Y_1^L + 1.5Y_2^L + 0.75t \le 1,\ 5.5Y_1^U + 2.5Y_2^U + 1.25t \ge 1\}. \tag{27}$$

Problem (27) is solved and the solution resulted is: $(Y_1^{*2L}, Y_1^{*2U}, Y_2^{*2L}, Y_2^{*2U}, t^{*2}) = (0.2116, 0.3937, 0, 0, 0.0635)$. Thus,

$[\tilde{X}^*]_\alpha^2 = (X_1^{*2L}, X_1^{*2U}, X_2^{*2L}, X_2^{*2U}) = \frac{1}{t^{*2}}(Y_1^{*2L}, Y_1^{*2U}, Y_2^{*2L}, Y_2^{*2U})$ $= (3.3333, 6.2, 0, 0)$.

It is the time to linearize $\bar{F}_1(X^L,X^U)$ and $\bar{F}_2(X^L,X^U)$ using the first-order Taylor series as follows:

$$\text{Linearization}\left(\bar{F}_1(X^L,X^U)\right) = [\underline{\text{Lin}}\,\bar{F}_1(X^L,X^U), \overline{\text{Lin}}\,\bar{F}_1(X^L,X^U)] = [-0.2655\ \ 0.0574](X_1^L - 3.3333) + [-0.2732\ \ 0](X_2^L - 1.4) + [0\ \ 0.0681](X_1^U - 3.3333) + [0\ \ 0.3336](X_2^U - 1.4) =$$
$$[-0.2655X_1^L - 0.2732X_2^L - 0.8854, 0.0574X_1^L + 0.0681X_1^U + 0.3336X_2^U + 1.2675]. \tag{28}$$

$$\text{Linearization}\left(\bar{F}_2(X^L,X^U)\right) = [\underline{\text{Lin}}\,\bar{F}_2(X^L,X^U), \overline{\text{Lin}}\,\bar{F}_2(X^L,X^U)] = [-0.8436\ 0.1839](X_1^L - 3.3333) + [-0.2812\ \ 0.0212]X_2^L + [-0.0954\ \ 0.4762](X_1^U - 6.2) + [-0.0433\ \ 0.0794]X_2^U =$$
$$[-0.8436X_1^L - 0.2812X_2^L - 0.0954X_1^U - 0.0433X_2^U - 3.5654, 0.1839X_1^L + 0.0212X_2^L + 0.4762X_1^U + 0.0794X_2^U + 3.4035]. \tag{29}$$

The membership functions are defined as follows:

$\mu_{\underline{\text{Lin}}\,\bar{F}_1} = -0.3489X_1^L - 0.359X_2^L + 2.1631$,

$\mu_{\overline{\text{Lin}}\,\bar{F}_1} = 0.0915X_1^L + 0.1085X_1^U + 0.5316X_2^U - 0.6666$,

and

$\mu_{\underline{\text{Lin}}\,\bar{F}_2} = -0.3134X_1^L - 0.1043X_2^L - 0.0354X_1^U + 3.4712$,

$\mu_{\overline{\text{Lin}}\,\bar{F}_2} = 0.0972X_1^L + 0.0112X_2^L + 0.2517X_1^U + 0.042X_2^U - 1.1628$.

Now, we form problem (21) for this example as follows:

$$\text{Maximize}\quad w_1\left(\mu_{\underline{\text{Lin}}\,\bar{F}_1} + \mu_{\overline{\text{Lin}}\,\bar{F}_1}\right) + w_2\left(\mu_{\underline{\text{Lin}}\,\bar{F}_2} + \mu_{\overline{\text{Lin}}\,\bar{F}_2}\right)$$
$$\text{s.t }\ (X^L,X^U) \in S_{\alpha=0.5}. \tag{30}$$

Problem (30) is solved for equal weights i.e., $w_1 = 0.5, w_2 = 0.5$ and the solution obtained is: $X^* = (X^{*L}, X^{*U}) = (X_1^{*L}, X_1^{*U}, X_2^{*L}, X_2^{*U}) = (3.3333, 4.24, 0, 1.4)$. Furthermore,

$\bar{F}_1(X^*) = [F_{l1}(X^*), F_{u1}(X^*)] = [-2.968, -0.4581]$,

$\bar{F}_2(X^*) = [F_{l2}(X^*), F_{u2}(X^*)] = [0.7719, 2.1302]$, where F_{li}, $F_{ui}, i = 1, 2$ are identified by the help of Remark 1 as follows:

$F_{l1}(X^L, X^U) = \frac{1.5X_2^L - 3.5X_1^U}{0.75X_1^L + 0.75X_2^L + 2.5}$, $F_{u1}(X^L, X^U) = \frac{-2.5X_1^L + 2.5X_2^U}{1.25X_1^U + 1.25X_2^U + 3.5}$,

And $F_{l2}(X^L, X^U) = \frac{6.5X_1^L + 0.75X_2^L}{5.5X_1^U + 2.5X_2^U + 1.25}$,

$F_{u2}(X^L, X^U) = \frac{7.5X_1^U + 1.25X_2^U}{4.5X_1^L + 1.5X_2^L + 0.75}$. In addition,

$\underset{X\in S}{\text{Maximize}}\, F_{l1} = -1.5813$, $\underset{X\in S}{\text{Maximize}}\, F_{u1} = -0.4581$,

$\underset{X\in S}{\text{Maximize}}\, F_{l2} = 1.14$, $\underset{X\in S}{\text{Maximize}}\, F_{u2} = 2.9524$

These values are obtained using the method of Charnes and Cooper and utilized to calculate the accuracy of a proposed solution X i.e., $Er(X)$ and $\epsilon(X)$.
Accordingly, it was calculated that: $\epsilon(X^*) = 0.8222$ and $Er(X^*) = 0.44$.
Furthermore;

Dfuz $([\tilde{X}^*]_{\alpha=0.5}) = (X_1^*, X_2^*) = (3.7866, 0.7)$,

Dfuz $\left([\tilde{F}_1(\tilde{X})]_\alpha\right) = -1.7131$, and

Dfuz $\left([\tilde{F}_2(\tilde{X})]_\alpha\right) = 2.9021$.

Comparison

In this section, we compare our results with the recent work of Arya et al. in which the outcomes are in the form of triangular fuzzy numbers:

$\tilde{X}_1 = (1.13, 3, 10.5)$, and

$\tilde{F}_1(\tilde{X}) = (-12.49, -0.625, 2.5334)$,

To make a comparison, the fuzzy numbers are changed into the intervals using their $\alpha - cuts$, for $\alpha = 0.5$. Thus, $X^{\text{Arya}} = (X^L, X^U) = (2.065, 6.75, 1.6875, 2.75)$,

$\overline{F}_1(X^{\text{Arya}}) = [-3.9692, 0.1114]$,

$\overline{F}_2(X^{\text{Arya}}) = [0.3246, 4.2996]$, and

$\epsilon(X^{\text{Arya}}) = 1.9$, $Er(X^{\text{Arya}}) = 0.95$.

Therefore, solution X^* proposed by this study is strongly preferred to X^{Arya} ($X^{\text{Arya}}\ PP\ X^*$) due to the fact that: $\epsilon(X^*) < \epsilon(X^{\text{Arya}})$ and $Er(X^*) < Er(X^{\text{Arya}})$.

EXAMPLE 2

$$\text{Maximize } \{\tilde{F}_1(\tilde{X}), \tilde{F}_2(\tilde{X}), \tilde{F}_3(\tilde{X})\} = \left\{\frac{-(2,\ 3,\ 4)\tilde{X}_1 + (1,\ 2,\ 3)\tilde{X}_2}{(0.5,\ 1,\ 1.5)\tilde{X}_1 + (0.5,\ 1,\ 1.5)\tilde{X}_2 + (2,\ 3,\ 4)}, \frac{(6,\ 7,\ 8)\tilde{X}_1 + (0.5,\ 1,\ 1.5)\tilde{X}_2}{(4,\ 5,\ 6)\tilde{X}_1 + (1,\ 2,\ 3)\tilde{X}_2 + (0.5,\ 1,\ 1.5)}, \frac{(0.5,\ 1,\ 1.5)\tilde{X}_1 + (3,\ 4,\ 5)\tilde{X}_2}{(1,\ 2,\ 3)\tilde{X}_1 + (2,\ 3,\ 4)\tilde{X}_2 + (1,\ 2,\ 3)}\right\}, \quad (31)$$

$$\text{s.t } -(0.5, 1, 1.5)\tilde{X}_1 + (0.5, 1, 1.5)\tilde{X}_2 \le -(0.5, 1, 1.5),$$
$$(1, 2, 3)\tilde{X}_1 + (2, 3, 4)\tilde{X}_2 \le (14, 15, 16),$$
$$-(0.5, 1, 1.5)\tilde{X}_1 \le -(2, 3, 4), -(0.5, 1, 1.5)\tilde{X}_1 - (8, 9, 10)\tilde{X}_2 \le -(8, 9, 10), \quad \tilde{X}_1, \tilde{X}_2 \ge 0.$$

In brief, setting $\alpha = 0.5$. and considering the interval arithmetic and ranking method 1 finally transform problem (31) into:

$$\text{Maximize } \{\overline{F}_1(X^L, X^U),\ \overline{F}_2(X^L, X^U),\ \overline{F}_3(X^L, X^U)\} = \left\{\frac{[1.5X_2^L - 3.5X_1^U, -2.5X_1^L + 2.5X_2^U]}{[0.75X_1^L + 0.75X_2^L + 2.5,\ 1.25X_1^U + 1.25X_2^U + 3.5]}, \frac{[6.5X_1^L + 0.75X_2^L,\ 7.5X_1^U + 1.25X_2^U]}{[4.5X_1^L + 1.5X_2^L + 0.75,\ 5.5X_1^U + 2.5X_2^U + 1.25]}, \frac{[0.75X_1^L + 3.5X_2^L,\ 1.25X_1^U + 4.5X_2^U]}{[1.5X_1^L + 2.5X_2^L + 1.5,\ 2.5X_1^U + 3.5X_2^U + 2.5]}\right\} \quad (32)$$

$$\text{s.t } S_{\alpha=0.5} = \{0.75X_2^L - 1.25X_1^U \le -1.25,$$
$$-0.75X_1^L + 1.25X_2^U \le -0.75,$$
$$1.5X_1^L + 2.5X_2^L \le 14.5, \quad 2.5X_1^U + 3.5X_2^U \le 15.5,$$
$$-0.75X_1^L \le -2.5, \quad -1.25X_1^U \le -3.5,$$
$$-1.25X_1^U - 9.5X_2^U \le -9.5,$$
$$-0.75X_1^L - 8.5X_2^L \le -8.5,$$
$$X_1^L \le X_1^U,\ X_2^L \le X_2^U, X_1^L, X_1^U, \quad X_2^L, X_2^U \ge 0\}.$$

For convenience, let $F = \{0.75Y_2^L - 1.25Y_1^U + 1.25t \le 0$, $-0.75Y_1^L + 1.25Y_2^U + 0.75t \le 0, 1.5Y_1^L + 2.5Y_2^L - 14.5t \le 0$, $2.5Y_1^U + 3.5Y_2^U - 15.5t \le 0$, $-0.75Y_1^L + 2.5t \le 0$, $-1.25Y_1^U + 3.5t \le 0$, $-1.25Y_1^U - 9.5Y_2^U + 9.5t \le 0$, $-0.75Y_1^L - 8.5Y_2^L + 8.5t \le 0$, $Y_1^L \le Y_1^U$, $Y_2^L \le Y_2^U$, $Y_1^L, Y_1^U, Y_2^L, Y_2^U, t \ge 0\}$.

In what follows to linearize $\overline{F}_i(X^L, X^U)$, $\underset{(X^L, X^U)\in S_{\alpha=0.5}}{\text{Maximize}}\ \overline{F}_i(X^L, X^U)$ is solved so as to find $[\tilde{X}^*]_\alpha^i = [X^{*iL}, X^{*iU}], i = 1, 2, 3$ .Let us set: $i = 1$. For this case, problem (12) is formulated as follows:

$$\text{Maximize } -2.5Y_1^L + 1.5Y_2^L - 3.5Y_1^U + 2.5Y_2^U \quad \text{s.t} \quad F \cup \{0.75Y_1^L + Y_2^L + 2.5t \le 1, Y_1^U + Y_2^U + 3.5t \ge 1\}. \tag{33}$$

Problem (33) is solved and the solution obtained is:

$(Y_1^{*1L}, Y_2^{*1L}, Y_1^{*1U}, Y_2^{*1U}, t^{*1}) = (0.4049, 0.4049, 0.17, 0.17, 0.1215)$.

Consequently, $[\tilde{X}^*]_\alpha^1 = (X_1^{*1L}, X_1^{*1U}, X_2^{*1L}, X_2^{*1U}) = \frac{1}{t^{*1}}(Y_1^{*1L}, Y_1^{*1U}, Y_2^{*1L}, Y_2^{*1U}) = (3.3333, 3.3333, 1.4, 1.4)$.

Let us set: $i = 2$. For this case, problem (12) is formed as follows:

$$\text{Maximize } 6.5Y_1^L + 0.75Y_2^L + 7.5Y_1^U + 1.25Y_2^U \quad \text{s.t } F \cup \{4.5Y_1^L + 1.5Y_2^L + 0.75t \le 1,\ 5.5Y_1^U + 2.5Y_2^U + 1.25t \ge 1\}. \tag{34}$$

Problem (34) is solved and the solution obtained is: $(Y_1^{*2L}, Y_2^{*2L}, Y_1^{*2U}, Y_2^{*2U}, t^{*2}) = (0.1983, 0.3101, 0.042, 0.042, 0.0595)$. Thus, $[\tilde{X}^*]_\alpha^2 = (X_1^{*2L}, X_1^{*2U}, X_2^{*2L}, X_2^{*2U}) = \frac{1}{t^{*2}}(Y_1^{*2L}, Y_1^{*2U}, Y_2^{*2L}, Y_2^{*2U}) = (3.3333, 5.2118, 0.7059, 0.7059)$.

Let us set: $i = 3$. For this case, problem (12) is formulated as follows:

$$\text{Maximize } 0.75Y_1^L + 3.5Y_2^L + 1.25Y_1^U + 4.5Y_2^U \quad \text{s.t } F \cup \{1.5Y_1^L + 2.5Y_2^L + 1.5t \le 1,\ 2.5Y_1^U + 3.5Y_2^U + 2.5t \ge 1\}. \tag{35}$$

Problem (35) is solved and the solution obtained is: $(Y_1^{*3L}, Y_2^{*3L}, Y_1^{*3U}, Y_2^{*3U}, t^{*3}) = (0.4033, 0.513, 0.0854, 0.1694, 0.121)$. Therefore, $[\tilde{X}^*]_\alpha^3 = (X_1^{*3L}, X_1^{*3U}, X_2^{*3L}, X_2^{*3U}) = \frac{1}{t^{*3}}(Y_1^{*3L}, Y_1^{*3U}, Y_2^{*3L}, Y_2^{*3U}) = (3.3333, 4.24, 0.7059, 1.4)$.

We linearize $\overline{F}_1(X^L, X^U)$, $\overline{F}_2(X^L, X^U)$, and $\overline{F}_3(X^L, X^U)$ using the first-order Taylor series as follows:

$$\begin{aligned} &\text{Linearization}\left(\overline{F}_1(X^L, X^U)\right) \\ &[\underline{\text{Lin}}\,\overline{F}_1(X^L, X^U), \overline{\text{Lin}}\,\overline{F}_1(X^L, X^U)] = \\ &[-0.2655\ \ 0.0574](X_1^L - 3.3333) + [-0.2732\ \ 0](X_2^L - 1.4) + \\ &[0\ \ 0.0681](X_1^U - 3.3333) + [0\ \ 0.3336](X_2^U - 1.4) = \\ &[-0.2655X_1^L - 0.2732X_2^L - 0.8854,\ 0.0574X_1^L + 0.0681X_1^U + 0.3336X_2^U + 1.2675]. \end{aligned} \tag{36}$$

$$\begin{aligned} &\text{Linearization}\left(\overline{F}_2(X^L, X^U)\right) = \\ &[\underline{\text{Lin}}\,\overline{F}_2(X^L, X^U), \overline{\text{Lin}}\,\overline{F}_2(X^L, X^U)] = \\ &[-0.6366 - 0.4152](X_1^L - 3.3333) + [-0.2122\ \ -0.1887](X_2^L - 0.7059) + \\ &[0.3045\ 0.4462](X_1^U - 5.2118) + [0.0351\ 0.0744](X_2^U - 0.7059) = \\ &[-0.6368X_1^L - 0.2122X_2^L + 0.3045X_1^U + 0.3045X_1^U + 0.0351X_2^U - 0.66, \\ &-0.4152X_1^L - 0.1872X_2^L + 0.4462X_1^U + 0.0744X_2^U + 0.8608] \end{aligned} \tag{37}$$

$$\begin{aligned} &\text{Linearization}\left(\overline{F}_3(X^L, X^U)\right) = \\ &[\underline{\text{Lin}}\,\overline{F}_3(X^L, X^U), \overline{\text{Lin}}\,\overline{F}_3(X^L, X^U)] = \\ &[-0.2547\ 0.0417](X_1^L - 3.3333) + [-0.4246\ 0.1944](X_2^L - 0.7059) + \\ &[-0.0384\ \ 0.1512](X_1^U - 4.24) + [-0.0537\ 0.5445](X_2^U - 1.4) = \\ &[-0.2547X_1^L - 0.4246X_2^L - 0.0384X_1^U - 0.0537X_2^U - 1.6796, \\ &0.0417X_1^L + 0.1944X_2^L + 0.1512X_1^U + 0.5445X_2^U + 1.3867] \end{aligned} \tag{38}$$

The membership functions are defined as follows:

$\mu_{\underline{\text{Lin}}\,\overline{F}_1} = -0.0597X_1^L - 0.0615X_2^L + 0.359,$

$\mu_{\overline{\text{Lin}}\,\overline{F}_1} = 0.1464X_1^L + 0.1737X_1^U + 0.851X_2^U - 1.6679,$

$\mu_{\underline{\text{Lin}}\,\overline{F}_2} = -0.1636X_1^L - 0.0545X_2^L - 0.0782X_1^U + 0.009X_2^U + 0.4911,$

$\mu_{\overline{\text{Lin}}\,\overline{F}_2} = -0.4443X_1^L - 0.2003X_2^L + 0.4474X_1^U + 0.0796X_2^U + 0.0778.$

$\mu_{\underline{\text{Lin}}\,\overline{F}_3} = -0.4122X_1^L - 0.6872X_2^L - 0.0621X_1^U - 0.0869X_2^U + 3.1275,$

$\mu_{\overline{\text{Lin}}\,\overline{F}_3} = 0.0505X_1^L + 0.2352X_2^L + 0.183X_1^U + 0.6589X_2^U - 1.4092.$

Finally, problem (21) is formed for this example as follows:

$$\text{Maximize } w_1\left(\mu_{\underline{\text{Lin}}\,\overline{F}_1}+\mu_{\overline{\text{Lin}}\,\overline{F}_1}\right)+w_2\left(\mu_{\underline{\text{Lin}}\,\overline{F}_2}+\mu_{\overline{\text{Lin}}\,\overline{F}_2}\right)+w_3\left(\mu_{\underline{\text{Lin}}\,\overline{F}_3}+\mu_{\overline{\text{Lin}}\,\overline{F}_3}\right)$$
$$\text{s.t } (X^L, X^U) \in S_{\alpha=0.5}. \quad (39)$$

Problem (39)is solved for equal weights i.e., $w_i = \frac{1}{3}, i = 1,2,3$ and the solution obtained is:
$X^* = (X^{*L}, X^{*U}) = (X_1^{*L}, X_1^{*U}, X_2^{*L}, X_2^{*U}) = (3.3333, 4.24, 0.7059, 1.4)$. Furthermore, the acceptable intervals for the value of the objective functions are:
$[\tilde{F}_1(X^*)]_{\alpha=0.5} = \overline{F}_1(X^*) = [F_{l1}(X^*), F_{u1}(X^*)] = [-2.4923, -0.4581]$, $[\tilde{F}_2(X^*)]_{\alpha=0.5} = \overline{F}_2(X^*) = [F_{l2}(X^*), F_{u2}(X^*)] = [0.7907, 1.9960]$, $[\tilde{F}_3(X^*)]_{\alpha=0.5} = \overline{F}_3(X^*) = [F_{l3}(X^*), F_{u3}(X^*)] = [0.2761, 1.4036]$,

where F_{li} and $F_{ui}, i = 1,2,3$ are determined considering remark 1 and are as follows:

$$F_{l1}(X^L, X^U) = \frac{1.5X_2^L - 3.5X_1^U}{0.75X_1^L + 0.75X_2^L + 2.5},$$
$$F_{u1}(X^L, X^U) = \frac{-2.5X_1^L + 2.5X_2^U}{1.25X_1^U + 1.25X_2^U + 3.5},$$
$$F_{l2}(X^L, X^U) = \frac{6.5X_1^L + 0.75X_2^L}{5.5X_1^U + 2.5X_2^U + 1.25},$$
$$F_{u2}(X^L, X^U) = \frac{7.5X_1^U + 1.25X_2^U}{4.5X_1^L + 1.5X_2^L + 0.75},$$
$$F_{l2}(X^L, X^U) = \frac{0.75X_1^L + 3.5X_2^L}{2.5X_1^U + 3.5X_2^U + 2.5},$$
$$F_{u2}(X^L, X^U) = \frac{1.25X_1^U + 4.5X_2^U}{1.5X_1^L + 2.5X_2^L + 1.5}.$$

In addition, $\epsilon(X^*) = 0.21$ and $Er(X^*) = 0.24$.

Moreover;
$\text{Dfuz}([\tilde{X}^*]_{\alpha=0.5}) = (X_1^*, X_2^*) = (3.7866,\ 1.053)$,
$\text{Dfuz}([\tilde{F}_1(\tilde{X})]_\alpha) = -1.4752$, $\text{Dfuz}([\tilde{F}_2(\tilde{X})]_\alpha) = 1.3933$,and
$\text{Dfuz}([\tilde{F}_3(\tilde{X})]_\alpha) = 0.8398$.

Comparison

In this section, we compare our results with the recent work ofAryaetal.inwhichtheoutcomesareintheformoftriangular fuzzy numbers $\tilde{X}_1 = (1.13, 3, 10.5)$, $\tilde{X}_2 = (1.375, 2, 3.5)$. In addition, the value of the objective functions are fuzzy numbers $\tilde{F}_1(\tilde{X}) = (-12.49, -0.625, 2.5334)$, $\tilde{F}_2(\tilde{X}) = (0.0995, 1.15, 13.956)$, and $\tilde{F}_3(\tilde{X}) = (0.0967, 0.7857, 5.379)$. Thus, to make a comparison, the fuzzy numbers are changed into the intervals using their $\alpha - cuts$, for $\alpha = 0.5$. Thus, $X^{\text{Arya}} = (X^L, X^U) = (2.065, 6.75, 1.6875, 2.75)$. Moreover, $\overline{F}_1(X^{\text{Arya}}) = [-3.6665\ 0.1114]$, $\overline{F}_2(X^{\text{Arya}}) = [0.3246\ \ 4.2996]$, $\overline{F}_3(X^{\text{Arya}}) = [0.2571\ \ 2.3607]$.

In addition, $\epsilon(X^{\text{Arya}}) = 0.95$, $Er(X^{\text{Arya}}) = 1.1$.

Therefore, solution X^* proposed by this study is strongly preferred to X^{Arya} ($X^{\text{Arya}}\ PP\ X^*$) due to the fact that:

$\epsilon(X^*) < \epsilon(X^{\text{Arya}})$ and $Er(X^*) < Er(X^{\text{Arya}})$.

The following were used to compute the accuracy of the solutions.
$\underset{X\in S}{\text{Maximize}}\, F_{l1} = -1.5813$, $\underset{X\in S}{\text{Maximize}}\, F_{u1} = -0.4581$,
$\underset{X\in S}{\text{Maximize}}\, F_{l2} = 1.1017$, $\underset{X\in S}{\text{Maximize}}\, F_{u2} = 2.378$,
$\underset{X\in S}{\text{Maximize}}\, F_{l3} = 0.4891$, $\underset{X\in S}{\text{Maximize}}\, F_{u3} = 1.4036$.

EXAMPLE 3 (REAL-WORLD APPLICATION)

In this section, a real world example which is a mathematical modelling of a university's plan to increase the rate of distance learning is considered.

$$\text{Maximize}\{\tilde{F}_1(\tilde{X}),\ \tilde{F}_2(\tilde{X})\} = \left\{\frac{(2,\ 3,\ 4)\tilde{X}_1 + (1,\ 3,\ 5)\tilde{X}_2}{(1,\ 2,\ 3)\tilde{X}_1 + (2,\ 3,\ 4)\tilde{X}_2 + (1,\ 2,\ 3)}, \frac{(1,\ 3,\ 5)\tilde{X}_1 + (4,\ 5,\ 6)\tilde{X}_2}{(1,\ 2,\ 3)\tilde{X}_1 + (2,\ 3,\ 4)\tilde{X}_2 + (2,\ 4,\ 6)}\right\} \quad (40)$$
$$\text{s.t } (2,\ 3,\ 4)\tilde{X}_1 + (1,\ 3,\ 5)\tilde{X}_2 \le (20,\ 25,\ 30),$$
$$(1,2,3)\tilde{X}_1 + (2,3,4)\tilde{X}_2 \le (5,10,15),\ \tilde{X}_1, \tilde{X}_2 \ge 0.$$

In brief, setting $\alpha = 0.5$ and considering the interval arithmetic and ranking method 1 finally transform problem (40) into:

$$\text{Maximize}\{\overline{F}_1(X^L, X^U),\ \overline{F}_2(X^L, X^U)\} = \left\{\overline{F}_1(X^L, X^U) = \frac{[2.5X_1^L + 2X_2^L,\ 3.5X_1^U + 3X_2^U]}{[1.5X_1^L + 2.5X_2^L + 1.5,\ 2.5X_1^U + 3.5X_2^U + 2.5]}, \overline{F}_2(X^L, X^U) = \frac{[2X_1^L + 4.5X_2^L,\ 3X_1^U + 5.5X_2^U]}{[1.5X_1^L + 2.5X_2^L + 3,\ 2.5X_1^U + 3.5X_2^U + 5]}\right\} \quad (41)$$
$$\text{s.t } S_{\alpha=0.5} = \{2.5X_1^L + 2X_2^L \le 22.5, 3.5X_1^U + 4X_2^U \le 27.5, 1.5X_1^L + 2.5X_2^L \le 7.5, 2.5X_1^U + 3.5X_2^U \le 12.5, X_1^L \le X_1^U, X_2^L \le X_2^U, X_1^L, X_1^U, X_2^L, X_2^U \ge 0\}.$$

For convenience, let:
$F = \{2.5Y_1^L + 2Y_2^L - 22.5t \le 0, 3.5Y_1^U + 4Y_2^U - 27.5t \le 0, 1.5Y_1^L + 2.5Y_2^L - 7.5t \le 0, 2.5Y_1^U + 3.5Y_2^U - 12.5t \le 0, Y_1^L \le Y_1^U, Y_2^L \le Y_2^U, Y_1^L, Y_1^U, Y_2^L, Y_2^U, t \ge 0\}$.
In what follows to linearize $\overline{F}_i(X^L, X^U)$, $\underset{(X^L,X^U)\in S_{\alpha=0.5}}{\text{Maximize}}\ \overline{F}_i(X^L, X^U)$ is solved so as to find $[\tilde{X}^*]_\alpha^i = [X^{*iL}, X^{*iU}], i = 1,2$ $[\tilde{X}^*]_\alpha^i = [X^{*iL}, X^{*iU}], i = 1,2$.

Let us set: $i = 1$. For this case, problem (12) is formulated as follows:

$$\begin{aligned} &\text{Maximize} \quad 2.5Y_1^L + 2Y_2^L + 3.5Y_1^U + 3Y_2^U \\ &\text{s.t } \mho \cup \{1.5Y_1^L + 2.5Y_2^L + 1.5t \le 1,\ 2.5Y_1^U \\ &\quad + 3.5Y_2^U + 2.5t \ge 1\}. \end{aligned} \tag{42}$$

Problem (42) is solved and the solution obtained is:
$(Y_1^{*1L}, Y_2^{*1L}, Y_1^{*1U}, Y_2^{*1U}, t^{*1}) = (0, 3.3333, 0, 0, 0.6667)$.
Therefore,
$[\tilde{X}^*]_\alpha^1 = (X_1^{*1L}, X_1^{*1U}, X_2^{*1L}, X_2^{*1U}) = \frac{1}{t^{*1}}(Y_1^{*1L}, Y_1^{*1U}, Y_2^{*1L}, Y_2^{*1U}) = (0, 5, 0, 0)$.
Let us set: $i = 2$. For this case, problem (12) is formulated as follows:

$$\begin{aligned} &\text{Maximize} \ \ 2Y_1^L + 4.5Y_2^L + 3Y_1^U + 5.5Y_2^U \\ &\text{s.t } \mho \cup \{1.5Y_1^L + 2.5Y_2^L + 3t \le 1,\ 2.5Y_1^U + \\ &\quad 3.5Y_2^U + 5t \ge 1\}. \end{aligned} \tag{43}$$

Problem (13) is solved and the solution obtained is:
$(Y_1^{*2L}, Y_2^{*2L}, Y_1^{*2U}, Y_2^{*2U}, t^{*2}) = (0, 0, 0, 1.1905, 0.3333)$.
Therefore,
$[\tilde{X}^*]_\alpha^2 = (X_1^{*2L}, X_1^{*2U}, X_2^{*2L}, X_2^{*2U}) = \frac{1}{t^{*2}}(Y_1^{*2L}, Y_1^{*2U}, Y_2^{*2L}, Y_2^{*2U}) = (0, 0, 0, 3.5714)$.

It is the time to linearize $\overline{F}_1(X^L, X^U)$, and $\overline{F}_2(X^L, X^U)$ using the first-order Taylor series as follows:

$$\begin{gathered} \text{Linearization}\left(\overline{F}_1(X^L, X^U)\right) = \\ \left[\underline{\text{Lin}}\,\overline{F}_1(X^L, X^U), \overline{\text{Lin}}\,\overline{F}_1(X^L, X^U)\right] = \\ [0\ \ 1]X_1^L + [0\ \ 0.8]X_2^L + \\ [-4\ \ 0.25](X_1^U - 5) + [-5.6\ \ 0.2143]X_2^U = \\ [-4X_1^U - 5.6X_2^U + 1.25,\ X_1^L + \\ 0.8X_2^L + 0.25X_1^U + 0.2143X_2^U + 20]. \end{gathered} \tag{44}$$

$$\begin{gathered} \text{Linearization}\left(\overline{F}_2(X^L, X^U)\right) = \\ \left[\underline{\text{Lin}}\,\overline{F}_2(X^L, X^U), \overline{\text{Lin}}\,\overline{F}_2(X^L, X^U)\right] = \\ [-3.2738\ \ 0.1143]X_1^L + [-5.4563\ \ 0.2571]X_2^L + \\ [0\ \ 1]X_1^U + [0\ \ 1.8333](X_2^U - 3.5714) = \\ [-3.2738X_1^L - 5.4563X_2^L - 6.5474, \\ 0.1143X_1^L + 0.2571X_2^L + X_1^U + 1.8333X_2^U] \end{gathered} \tag{45}$$

The membership functions are defined as follows:
$\mu_{\underline{\text{Lin}}\,\overline{F}_1} = -0.2X_1^U - 0.28X_2^U + 1.25,$
$\mu_{\overline{\text{Lin}}\,\overline{F}_1} = 0.16X_1^L + 0.128X_2^L + 0.04X_1^U + 0.0343X_2^U,$
$\mu_{\underline{\text{Lin}}\,\overline{F}_2} = -0.2X_1^L - 0.3333X_2^L + 1,$
$\mu_{\overline{\text{Lin}}\,\overline{F}_2} = 0.0156X_1^L + 0.0351X_2^L + 0.1366X_1^U + 0.2505X_2^U.$

Finally, problem (21) is formed for this example as follows:

$$\begin{gathered} \text{Maximize} \\ w_1\left(\mu_{\underline{\text{Lin}}\,\overline{F}_1} + \mu_{\overline{\text{Lin}}\,\overline{F}_1}\right) + w_2\left(\mu_{\underline{\text{Lin}}\,\overline{F}_2} + \mu_{\overline{\text{Lin}}\,\overline{F}_2}\right) \\ \text{s.t } (X^L, X^U) \in S_{\alpha=0.5}. \end{gathered} \tag{46}$$

Problem (48) is solved for equal weights i.e., $w_i = 0.5, i = 1, 2$ and the solution obtained is:

$X^* = (X^{*L}, X^{*U}) = (X_1^{*L}, X_1^{*U}, X_2^{*L}, X_2^{*U}) = (0, 0, 0, 3.5714).$
Furthermore, the acceptable intervals for the value of the objective functions are:
$[\tilde{F}_1(X^*)]_{\alpha=0.5} = \overline{F}_1(X^*) = [F_{l1}(X^*), F_{u1}(X^*)] = [0,\ 7.1428],$
$[\tilde{F}_2(X^*)]_{\alpha=0.5} = \overline{F}_2(X^*) = [F_{l2}(X^*), F_{u2}(X^*)] = [0,\ 6.5476].$
where F_{li} and $F_{ui}, i = 1, 2$ are determined considering remark 1 and are as follows:

$$F_{l1}(X^L, X^U) = \frac{2.5X_1^L + 2X_2^L}{2.5X_1^U + 3.5X_2^U + 2.5}, \quad F_{u1}(X^L, X^U) = \frac{3.5X_1^U + 3X_2^U}{1.5X_1^L + 2.5X_2^L + 1.5},$$

$$F_{l2}(X^L, X^U) = \frac{2X_1^L + 4.5X_2^L}{2.5X_1^U + 3.5X_2^U + 5}, \quad F_{u2}(X^L, X^U) = \frac{3X_1^U + 5.5X_2^U}{1.5X_1^L + 2.5X_2^L + 3},$$

In addition, $\epsilon(X^*) = 0.8$ and $Er(X^*) = 0.6$.

Comparison

In this section, we compare our results with the recent work of Arya et al. in which the outcomes are in the form of triangular fuzzy numbers $\tilde{X}_1 = (0, 5.14, 5.14)$, $\tilde{X}_2 = (0, 0, 3.86)$. Thus, to make a comparison, the fuzzy numbers are changed into the intervals using their $\alpha - cuts$, for $\alpha = 0.5$. Thus, $X^{\text{Arya}} = (X^L, X^U) = (0, 3.86, 2.57, 5.17)$.
Moreover,
$\overline{F}_1(X^{\text{Arya}}) = [0.1699, 3.6618]$, $\overline{F}_2(X^{\text{Arya}}) = [0.3532, 4.2456]$.
In addition, $\epsilon(X^{\text{Arya}}) = 2.3$, $Er(X^{\text{Arya}}) = 0.61$.
Therefore, solution X^* proposed by this study is strongly preferred to X^{Arya} due to the fact that:
$\epsilon(X^*) < \epsilon(X^{\text{Arya}})$ and $Er(X^*) < Er(X^{\text{Arya}})$.
The following were utilized to compute the accuracy of the solutions.
$\underset{X\in S}{\text{Maximize}}\, F_{l1} = 0.8333$, $\underset{X\in S}{\text{Maximize}}\, F_{u1} = 11.6667$,
$\underset{X\in S}{\text{Maximize}}\, F_{l2} = 0.871$, $\underset{X\in S}{\text{Maximize}}\, F_{u2} = 6.5476$.

CONCLUSION

In this paper, a method was introduced in two stages to address fully fuzzy multi-objective linear fractional programming problem (FFMOLFPP). Applying this method finally transforms FFMOLFPP into linear programming problem (LPP). To construct the approach, the notion of $\alpha - cuts$, variable transformations, the first-order Taylor expansion, the weighted sum approach, and the membership functions are used. The solution resulted at the end of the algorithm is at least a weakly ϵ −efficient solution for the main problem. Three examples were taken from Arya et al. and the results demonstrated that our approach could come out with more accurate solutions than the reference. This should be mentioned that this method is applicable for FFMOLFPP with any kind of fuzzy coefficients, while the method of Arya et al. is only designed for triangular fuzzy numbers.

ACKNOWLEDGEMENTS

This research was funded by Universiti Kebangsaan Malaysia through Grant No. UKM, ST-2019-016. Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

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