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Hardy's Inequality for Functions of Several Complex Variables

(Ketidaksamaan Hardy untuk Fungsi Beberapa Pemboleh Ubah Kompleks)

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ABSTRACT

We obtain a generalization of Hardy's inequality for functions in the Hardy space $H^1(\mathbb{B}_d)$, where $\mathbb{B}_d$ is the unit ball $\{z = (z_1, \ldots, z_d) \in \mathbb{C}^d \,|\, \Sigma_{i=1}^d |z_i|^2 < 1\}$. In particular, we construct a function ϕ on the set of d –dimensional multi-indices $\{n = (n_1, \ldots, n_d) \mid n_i \in \mathbb{N} \cup \{0\}\}$ and prove that if $f(z) = \Sigma\, a_n z^n$ is a function in $H^1(\mathbb{B}_d)$, then $\sum_{|n|=0}^{\infty} \frac{|a_n|}{\phi(n)+1} \leq \pi \|f\|_1$. Moreover, our proof shows that this inequality is also valid for functions in Hardy space on the polydisk $H^1(\mathbb{B}^d)$.

Keywords: Hardy's inequality; Hardy space and Hilbert's inequality

ABSTRAK

Kami memperoleh generalisasi ketidakseimbangan Hardy's untuk fungsi dalam ruang Hardy $H^1(\mathbb{B}_d)$, dengan $\mathbb{B}_d$ adalah unit bola $\{z = (z_1, \ldots, z_d) \in \mathbb{C}^d \,|\, \Sigma_{i=1}^d |z_i|^2 < 1\}$. Secara khususnya, kami membina fungsi ϕ pada set indeks pelbagai d –dimensi $\{n = (n_1, \ldots, n_d) \mid n_i \in \mathbb{N} \cup \{0\}\}$ dan membuktikan bahawa jika $f(z) = \Sigma\, a_n z^n$ adalah fungsi di $H^1(\mathbb{B}_d)$, kemudian $\sum_{|n|=0}^{\infty} \frac{|a_n|}{\phi(n)+1} \leq \pi \|f\|_1$. Selain itu, bukti kami menunjukkan bahawa ketidakseimbangan ini juga adalah sah untuk fungsi dalam ruang Hardy ke atas polidisk $H^1(\mathbb{B}^d)$.

Kata kunci: Ketidaksamaan Hardy; Ruang Hardy dan ketidaksamaan Hilbert

INTRODUCTION

For $z = (z_1, \ldots, z_d)$ in the d –dimensional complex Euclidean space $\mathbb{C}^d$, we define $\|z\|^2 = \sum_{i=1}^d |z_i|^2$. Let $\mathbb{B}_d$ denote the open unit ball containing $z \in \mathbb{C}^d$ such that $\|z\|^2 < 1$. A function $f : \mathbb{B}_d \to \mathbb{C}$ is holomorphic if for each $i = 1, \ldots, d$ and each fixed $z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_d$, the function $f_i : \xi \mapsto f(z_1, \ldots, z_{i-1}, \xi, z_{i+1}, \ldots, z_d)$ is holomorphic as a function of one variable. For $0 < p < \infty$, the Hardy space $H^p(\mathbb{B}_d)$ consists of all holomorphic functions f defined $\mathbb{B}_d$ on satisfying

$$\|f\|_p^p = sup_{0<r<1} \int_{\mathbb{S}_d} |f(r\xi)|^p \, d\sigma(\xi) < \infty,$$

where $\mathbb{S}_d$ is the boundary of $\mathbb{B}_d$ and $d\sigma$ is the normalized surface measure. Note that one can also define Hardy space of functions defined on the polydisk $\mathbb{B}^d = \mathbb{B} \times \ldots \times \mathbb{B}$ as the space of holomorphic functions f satisfying:

$$sup_{0<r<1} \int_{[0,2\pi]^d} |f(re^{i\theta})|^p \frac{d\theta}{(2\pi)^d} < \infty,$$

where $e^{i\theta} = (e^{i\theta_1}, \ldots, e^{i\theta_d})$ and $d\theta = d\theta_1 \ldots d\theta_d$.

For the case $p = 1$, Hardy's inequality for functions of one variable defined on the unit ball in $\mathbb{C}$ is well-known. It states that if $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H^1$, then

$$\sum_{n=0}^{\infty} \frac{|a_n|}{n+1} \leq \pi \|f\|_1,$$

see Duren (1970).

There are some connections between Hardy's inequality and inequalities in other Hilbert spaces. For example, Zhu (2004) translated this Hardy's inequality to the inequality $\sum_{n=0}^{\infty} \frac{\Gamma(\alpha+2)\Gamma\left(\frac{\pi}{2}+1\right)}{(n+1)\Gamma\left(\frac{\pi}{2}+\alpha+2\right)} |a_n| \leq \pi \int_{\mathbb{B}} |f(z)| \, dA_\alpha(z)$ for functions $f = \sum_{n=1}^{\infty} a_n z^n$ in the Bergman space A_α^1.

Sometimes, Hardy's inequality appears in an integral form. In Sababheh (2008a, 2008b), the author proved Hardy-type inequalities concerning the integral of the Fourier transform $\hat{f}$ of a function f with certain properties.

That is, for $f \in L^1(\mathbb{R})$ with $\int_{-\infty}^{x} f(t)\,dt \in L^1(\mathbb{R})$, $\hat{f}(\xi) = 0$ when $\xi < 0$ and $\alpha > 2$, the inequality $\int_0^{\infty} \frac{|\hat{f}(\xi)|^{\alpha}}{\xi} d\xi \leq 2\pi \|f\|^{\alpha}$ holds.

There is also a generalization on the multiplier $\frac{1}{n+1}$ in the summation of Hardy's inequality. Paulsen and Singh (2015) replaced the term $\frac{1}{n+1}$ in $\sum_{n=0}^{\infty} \frac{|a_n|}{n+1} \leq \pi \|f\|_1$ by a larger class of sequences. They proved that there is a constant A therefore if (c_n) is a sequence in some specific sequence space, then for any f in Hardy space H^1, the inequality holds $\sum_{n=0}^{\infty} |c_n a_n| \leq A \|c_n\| \|f\|_1$. This generalization is for a function f of one variable.

To generalize Hardy's inequality to functions f of several complex variables, we need to concern whether our functions are defined on the unit ball $\mathbb{B}_d$ or the polydisk $\mathbb{B}^d$. Basically, we cannot apply iterated integrals (d–times) to a function in $H^p(\mathbb{B}_d)$ as we usually do to functions in $H^p(\mathbb{B}^d)$.

In this paper, we will show that we can adjust to the proof in Duren (1970) to obtain Hardy's inequality that is valid for functions in either $H^p(\mathbb{B}_d)$ or $H^p(\mathbb{B}^d)$. A difficulty in the case of functions of several complex variables is that a holomorphic function f is represented by $f(z) = \Sigma a_n z^n$ where $n = (n_1, \ldots, n_d)$ is a multi-index. However, we will show that the set of multi-indices can be totally ordered in the way which enables us to prove a generalized Hardy's inequality.

Main Theorems

For multi-indices $n = (n_1, \ldots, n_d)$ and $m = (m_1, \ldots, m_d)$ in $\mathbb{N}_0^d$ where $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$, we define $|n| = \sum_{i=1}^{d} n_i$, $n! = n_1! \ldots n_d!$ and $n \pm m = (n_1 \pm m_1, \ldots, n_d \pm m_d)$. For $z = (z_1, \ldots, z_d) \in \mathbb{C}^d$, we also define $z^n = z_1^{n_d} \ldots z_d^{n_d}$. With this notation and for a given $k \in \mathbb{N}_0$, there are $\frac{(k+d-1)!}{k!(d-1)!}$ terms of $a_n z^n$ when $|n| = k$.

First we consider a lemma by Peter Duren which defines a bilinear form on vectors $x = (x_n)$ and $y = (y_n)$ in $\mathbb{C}^N$ and prove that it is bounded. We then generalize this result to the case where n are multi-indices. This result plays an important role in proving the Hilbert's inequality in Lemma 3 and Hardy's inequality in Theorem 2.

Lemma 1. Let $\psi \in L^{\infty}([0, 2\pi])$ and $\lambda_n = \frac{1}{2\pi}\int_0^{2\pi} e^{-int}\psi(t)\,dt$, $n = 0, 1, 2, \ldots$. Let $x = (x_1, \ldots, x_N)$ and $y = (y_1, \ldots, y_N)$ be vectors in $\mathbb{C}^N$. Define

$$A_N(x, y) = \sum_{n,m=0}^{N} \lambda_{n+m} x_n y_m.$$

Then

$$|A_N(x, y)| \leq \|\psi\|_{\infty} \|x\| \|y\|.$$

Proof. This proof is due to Duren (1970),

$$\begin{aligned}
|A_N(x, x)| &= \left| \sum_{n,m=0}^{N} \lambda_{n+m} x_n x_m \right| \\
&= \left| \frac{1}{2\pi}\int_0^{2\pi} \sum_{n,m=0}^{N} e^{-int} x_n \cdot e^{-imt} x_m \psi(t)\,dt \right| \\
&= \left| \frac{1}{2\pi}\int_0^{2\pi} \left(\sum_{n=0}^{N} e^{-int} x_n \right)^2 \psi(t)\,dt \right| \\
&\leq \frac{1}{2\pi}\int_0^{2\pi} \left| \sum_{n=0}^{N} e^{-int} x_n \right|^2 \|\psi\|_{\infty}\,dt \\
&= \frac{\|\psi\|_{\infty}}{2\pi}\int_0^{2\pi} \left| \sum_{n=0}^{N} e^{-int} x_n \right|^2 dt.
\end{aligned}$$

Now, we obtain,

$$\left| \sum_{n=0}^{N} e^{-int} x_n \right|^2 = \left(\sum_{n=0}^{N} e^{-int} x_n \right)\left(\sum_{n=0}^{N} \overline{e^{-\mathrm{int}} x_n} \right) = \sum_{n=0}^{N} |x_n|^2 + \sum_{n \neq m} e^{i(m-n)t} x_n \overline{x_m}. \tag{1}$$

Note that,

$$\int_0^{2\pi} e^{ikt}\,dt = 0$$

if $k \neq 0$. Therefore,

$$|A_N(x,x)| \leq \frac{\|\psi\|_{\infty}}{2\pi}\int_0^{2\pi} \left| \sum_{n=0}^{N} e^{-int} x_n \right|^2 dt = \frac{\|\psi\|_{\infty}}{2\pi} 2\pi \|x\|^2 = \|\psi\|_{\infty} \|x\|^2. \tag{2}$$

This bilinear form also satisfies the polarization identity

$$A_N(x, y) = \tfrac{1}{4} A_N(x + y, x + y) - \tfrac{1}{4} A_N(x - y, x - y).$$

Then by the parallelogram law,

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2,$$

we obtain

$$\begin{aligned}
|A_N(x, y)| &= \tfrac{1}{4} |A_N(x + y, x + y) - A_N(x - y, x - y)| \\
&\leq \tfrac{1}{4} (|A_N(x + y, x + y)| + |A_N(x - y, x - y)|) \\
&= \tfrac{1}{4} \|(\psi\|_{\infty} (\|x + y\|^2 + \|x - y\|^2) = \tfrac{1}{2} \|\psi\|_{\infty}(\|x\|^2 + \|y\|^2).
\end{aligned}$$

We can see that when $\|x\| = \|y\| = 1$, $|A_N(x, y)| \leq \|\psi\|_{\infty}$.

Hence $|A_N(x, y)| \leq \|\psi\|_{\infty} \|x\| \|y\|$.

To generalize this inequality to the case of multi-indices $n = (n_1, \ldots, n_d)$, we need a new formula for λ_n. Then, if we can find an upper bound of $|A_N(x, x)|$ in terms of $\|x\|$, we automatically obtain an upper bound of $|A_N(x, y)|$. The reason is that the rest of this proof depends only on properties of the norm.

Now, let $x = (x_n)$ be a vector where n is a multi-index with $|n| = 0, 1, \ldots, N$. For example, if $N = 3$ and $d = 2$, then a vector (x_n) could be $(x_{00}, x_{01}, x_{10}, x_{11}, x_{20}, x_{12}, x_{21}, x_{30}, x_{03})$. Note that we can consider (x_n) when $0 \le |n| \le N$ as a finite sequence with $\sum_{k=0}^{N} \frac{(k+d-1)!}{k!(d-1)!}$ terms. We also define $\|x\|$ to be $\sqrt{\sum_{|n|}^{N} |x_n|^2}$.

Now, when n is a multi-index, the term e^{-int} in the formula for λ_n is invalid. For the first try, one may replace n by $|n|$ and let $\lambda_n = \frac{1}{2\pi}\int_0^{2\pi} e^{-i|n|t}\psi(t)\,dt$. Unfortunately, the map $|\,.\,| : n \mapsto |n|$ is not injective. There exist multi-indices r, s such that $r \ne s$ but $|r| = |s|$, for example

$$|(1, 0, 0, \ldots, 0)| = |(0, 1, \ldots, 0)|.$$

Therefore,

$$\frac{1}{2\pi}\int_0^{2\pi} \sum_{n \ne m} e^{i(|n|-|m|)t} x_n \overline{x_m}\, dt \ne 0$$

and hence

$$\frac{1}{2\pi}\int_0^{2\pi} \left| \sum_{n=0}^{N} e^{-int} x_n \right|^2 dt \ne \|x\|^2.$$

Thus we will not obtain an analogue of inequality (2). However, it suggests that if we have an injective function $\phi(n)$ on the set of multi-indices and let

$$\lambda_n = \frac{1}{2\pi}\int_0^{2\pi} e^{-i\phi(n)t}\psi(t)\,dt,$$

then the proof of Lemma 1 will also be valid for the case where n is a multi-index. This will lead to Lemma 2 below.

Before we state Lemma 2, let us discuss the existence of ϕ. We know from the Zermelo's well-ordering theorem that every set can be well-ordered (and hence totally ordered) which implies the existence of an injective function ϕ from any set to the set $\mathbb{N}$. However, the proof of the Zermelo's well-ordering theorem is non-constructive. Below, we s` give an explicit construction of an injective function ϕ from the set of all multi-indices to the set $\mathbb{N}$.

The following Lemma 2 and Lemma 3 hold for an arbitrary injective function ϕ. However Theorem 1 requires that ϕ has to be independent of the order N of a multi-index n. This is because, in the proof of Theorem 1, we find an upper bound of the summation $\sum_{|n|=0}^{N} \lambda_n |a_n|$, in the following Inequality (3). Then we take $N \to \infty$ to obtain an upper bound of $\sum_{|n|=0}^{\infty} \lambda_n |a_n|$. This strategy suggests that λ_n must be independent of N.

We first look for a function ϕ defined on the set of multi-indices which is independent of N. Consider the relation $\le$ for multi-index notation. We say that $n \le m$ if $n_i \le m_i$ for all i. This relation is merely partially ordered and, for example, we cannot compare (1, 0, 1) and (0, 1, 0). Now, for $n \ne m$, we denote $n \prec m$ if,

1. $|n| < |m|$ or
2. $|n| = |m|$ with

$$n_1 n_2 \ldots n_{d_{|n|+1}} < m_1 m_2 \ldots m_{d_{|n|+1}},$$

where $n_1 n_2 \ldots n_{d_{|n|+1}}$ is the representation of a number in base $|n| + 1$.

Precisely,

$$n_1 n_2 \ldots n_{d_{|n|+1}} = \sum_{k=1}^{d} n_k (|n|+1)^{d-k}.$$

Now, the relation $\prec$ is totally ordered. Since the relation $\prec$ is totally ordered, we can construct an injective function ϕ defined according $\prec$ to as follows.

For example, when $d = 3$, we have $(0, 0, 0) \prec (0, 0, 1) \prec (0, 1, 0) \prec (1, 0, 0) \prec (0, 0, 2) \prec (0, 1, 1) \prec (0, 2, 0) \prec (1, 0, 1) \prec (1, 1, 0) \prec (2, 0, 0) \prec (0, 0, 3) \prec \ldots$.

It is easy to see that we arrange the multi-indices n according to their order $|n|$. Then, among multi-indices with the same order, we arrange them according to their values in base $|n| + 1$, each of which is a unique representation.

Then we define $\phi(n)$ according to the arrangement of n via the relation $\prec$. As in this example, we obtain $\phi((0, 0, 0)) = 0$, $\phi((0, 0, 1)) = 1$, $\phi((0, 1, 0) = 2$, $\phi((1, 0, 0)) = 3$, $\phi((0, 0, 2) = 4, \ldots$. We note that ϕ is injective. When $d = 1$, we also have $\phi(n) = n$. We now generalize Lemma 1 to the following lemma for vectors (x_n) and (y_n) where n is a multi-index.

Lemma 2. Let $\psi \in L^\infty([0, 2\pi])$, $N \in \mathbb{N}$, $N = \{ n = (n_1, \ldots, n_d) : 0 \le |n| \le N$, and ϕ be an injective function from N to $\mathbb{N}$. Let

$$\lambda_n = \frac{1}{2\pi}\int_0^{2\pi} e^{-i\phi(n)t}\psi(t)\,dt,$$

$n = (n_1, \ldots, n_d) \in \mathbb{N}_0^d$ and $A_N(x, y) = \sum_{|n|,|m|=0}^{N} \lambda_{n+m} x_n y_m$. Then $|A_N(x, y)| \le \|\psi\|_\infty \|x\| \|y\|$.

Proof. The proof of this lemma is analogous to that of Lemma 1.

In the previous lemmas, the function ψ is an arbitrary function in $L^\infty([0, 2\pi])$ and Lemma 2 is valid for any injective function ϕ. Next, in Lemma 3 (and also later in Theorem 2), we will choose a specific function ψ, i.e. we will use $\psi(t) = ie^{-it}(\pi - t)$. This will fix $\|\psi\|_\infty$ and thus a constant in the equality. With this specific choice of ψ together with an injective function ϕ, we define λ_{n+m} and compute $|\lambda_{n+m}|$, as well as $\|\psi\|_\infty$. Applying this result to the inequality in Lemma 2, we obtain another version of Hilbert's inequality.

Lemma 3. Let $N \in \mathbb{N}$,

$\mathcal{N} = \{n = (n_1, \ldots, n_d) : 0 \le |n| \le N\}$, and ϕ be an injective function from $\mathcal{N}$ to $\mathbb{N}$. Then

$$\left|\sum_{|n|,|m|=0}^{N} \frac{x_n y_m}{\phi(n+m)+1}\right| \le \pi\|x\|\|y\|.$$

Proof. Choose $\psi(t) = ie^{-it}(\pi - t)$. By Lemma 2, we have λ_{n+m}

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{-i\phi(n+m)t} ie^{-it}(\pi - t)\, dt$$

$$= \frac{1}{2\pi} i \int_0^{2\pi} e^{-i(\phi(n+m)+1)t}(\pi - t)\, dt$$

$$= \frac{1}{2\pi} i(\pi\int_0^{2\pi} e^{-i(\phi(n+m)+1)t} dt - \int_0^{2\pi} e^{-i(\phi(n+m)+1)t} t\, dt).$$

By the Euler formula

$$e^{ix} = \cos x + i \sin x,$$

the first integral can be eliminated and the second integral can be decomposed as

$$\int_0^{2\pi} t\cos\left[(\phi(n+m)+1)t\right] dt + i\int_0^{2\pi} t\sin\left[(\phi(n+m)+1)t\right] dt.$$

Using integration by parts, we also obtain

$$\int_0^{2\pi} t\cos\left[(\phi(n+m)+1)t\right] dt = 0.$$

However,

$$\int_0^{2\pi} t\sin\left[(\phi(n+m)+1)t\right] dt = -\frac{2\pi}{\phi(n+m)+1}.$$

Therefore, $|\lambda_{n+m}| = \frac{1}{\phi(n+m)+1}$. Consider $|\psi(t)| = |ie^{-it}(\pi - t)| = |\pi - t)| \le \pi$, for all t. Therefore, $\|\psi|_\infty \le \pi$. Then, by Lemma 2,

$$\left|\sum_{|n|,|m|=0}^{N} \frac{x_n y_m}{\phi(n+m)+1}\right| \le \pi\|x\|\|y\|.$$

Now we shall consider a function $f \in H^1(\mathbb{B}_d)$. Suppose that the Taylor expansion of f is of the form $f(z) = \Sigma\, a_n z^n$. Then, by orthogonality of $\{z^n\}$ as functions in H^2, we can compute the norm of f in terms of the sum of the square of the Taylor coefficients $\Sigma\, |a_n|^2$. The next theorem shows a relation between a weighted sum of coefficients in the Taylor expansion of $f \in H^1$ and the norm $\|f\|_1$.

Theorem 1. Let

$f(z) = \sum_{|n|=0}^{\infty} a_n z^n \in H^1$ and $\lambda_n \ge 0$. Then $\sum_{|n|=0}^{\infty} \lambda_n |a_n| \le \|\psi\|_\infty \|f\|_1$.

Proof. Since $f \in H^1$, there exist g and h in the same H^2 such that $f = gh$ and $\|g\|_2^2 = \|h\|_2^2 = \|f\|_1$. We can also write g and h as $g(z) = \sum_{|n|=0}^{\infty} b_n z^n$ and $h(z) = \sum_{|n|=0}^{\infty} c_n z^n$. Consider,

$f(z) = \Sigma\, a_n z^n = (\Sigma b_n z^n)(\Sigma c_n z^n)$.

For the case $d = 1$, it is easy to verify that $a_n = \sum_{|n|=0}^{\infty} b_k c_{n-k}$. For $d \ge 1$, the product $b_k z^k c_s z^s$ is of the form

$$b_{k_1,\ldots,k_d} c_{s_1,\ldots,s_d} z_1^{k_1+s_1} \ldots z_d^{k_d+s_d}.$$

Therefore, to obtain

$$a_n z^n = a_{n_1,\ldots,n_d} z_1^{n_1} \ldots z_d^{n_d},$$

we need all possible choices of $k = (k_1, \ldots, k_d)$ and $s = (s_1, \ldots, s_d)$ such that $s = n - k$, which is the same as in the case $d = 1$. Therefore, we also obtain $a_n = \Sigma_{0\le k\le n} b_k c_{n-k}$. However, we should note that, for the case $d = 1$, there are $n + 1$ terms in $a_n = \Sigma_{k=0} b_k c_{n-k}$ whereas there are $(n_1 + 1) . (n_2 + 1) \ldots (n_d + 1)$ terms in $a_n = \Sigma_{0\le k\le n} b_k c_{n-k}$ for the case $d \ge 1$. Then, by the triangle inequality, we have

$$\sum_{|n|=0}^{N} \lambda_n |a_n| = \sum_{|n|=0}^{N} \lambda_n \left|\sum_{0\le k\le n} b_k c_{n-k}\right|$$

$$\le \sum_{|n|=0}^{N} \lambda_n \sum_{0\le k\le n} |b_k||c_{n-k}|.$$

The summation $\Sigma_{0\le k\le n} |b_k|\, |C_{n-k}|$ depends on n. Therefore,

$$\sum_{|n|=0}^{N} \lambda_n \sum_{0\le k\le n} |b_k||c_{n-k}| \le \sum_{|k|,|m|=0}^{N} \lambda_{k+m} |b_k||c_m| \le \|\psi\|_\infty \|g\|_2 \|h\|_2.$$

The last inequality is a consequence of Lemma 2. Since $\|g\|_2\, \|h\|_2 = \|f\|_1$, we obtain

$$\sum_{|n|=0}^{N} \lambda_n |a_n| \le \|\psi\|_\infty \|f\|_1. \tag{3}$$

for any N. By letting $N \to \infty$, we obtain $\sum_{|n|=0}^{\infty} \lambda_n |a_n| \le \|\psi\|_\infty \|f\|_1$.

Next, we will show that the Hardy's inequality for functions of several complex variables can be easily proved by using Theorem 1 together with the function ψ defined in Lemma 3.

Theorem 2. If $f(z) = \sum_{|n|=0}^{\infty} a_n z^n \in H^1$ and ϕ is an injective function from the set of multi-indices to the set $\mathbb{N}$, then

$$\sum_{|n|=0}^{\infty} \frac{|a_n|}{\phi(n)+1} \leq \pi \|f\|_1, \tag{4}$$

Proof. Let f be any function in H^1 and $\psi(t) = ie^{-it}(\pi - t)$ for $0 \leq t \leq 2\pi$. Then $\|\psi\|_\infty \leq \pi$. As in the proof of Lemma 3, we obtain $|\lambda_n| = \frac{1}{\phi(n)+1}$. Then Inequality (4) follows from Theorem 1. We shall also call Inequality (4) Hardy's inequality.

Discussion

Our Hardy's inequality (4) comes directly from Inequality (3) in Theorem 1. With our specific choice of function ψ, we have $\|\psi\|_\infty \leq \pi$ and $|\lambda_n| = \frac{1}{\phi(n)+1}$. The latter holds for any injective function ϕ which is independent of f. The proof of Theorem 1 involves only the coefficients of the Taylor expansion of f, regardless of where f is defined. Therefore, Hardy's inequality (Inequality (4)) holds for all functions f in Hardy space $H^1(\mathbb{B}_d)$ as well as functions f in $H^1(\mathbb{B}^d)$.

Let us note that Lemmas 2 and 3 are true for any injective function ϕ defined on the set of multi-indices n when $0 \leq |n| \leq N$ and they do not require that ϕ has to be N–independent. For example, let us consider a function Φ defined by

$$\Phi(n_1, n_2, \ldots, n_d)) = n_1 n_2 \ldots n_{d_{N+1}},$$

which is also injective but less complicated than the function φ we constructed earlier. The proof of Lemma 3 is also true for this function Φ. However, we cannot use this Φ in Theorem 1 because the formula for Φ depends on N which will cause a problem when we take $N \to \infty$.

The proof of Theorem 2 is valid for any injective function from the set of multi-indices to the set $\mathbb{N}$. Our specific example ϕ (constructed before Lemma 2) has a property that $\phi(n) = n$ when $d = 1$. Thus Inequality (4) reduces to Hardy's inequality $\sum_{n=0}^{\infty} \frac{|a_n|}{n+1}$ when $d = 1$. Suppose that ϕ is another injective function such that when $d = 1$, the value $\phi(n)$ is not necessarily equal to n. Then, Inequality (4) will yield another version of Hardy's inequality for $d = 1$, where the denominators $n+1$ of the summation $\sum_{n=0}^{\infty} \frac{|a_n|}{n+1}$ in the standard Hardy's inequality will be replaced by a sequence of distinct integers greater than 1. Therefore, not only that Inequality (4) generalizes the standard Hardy's inequality to the case $d > 1$, it also gives a generalization in the case $d = 1$.

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